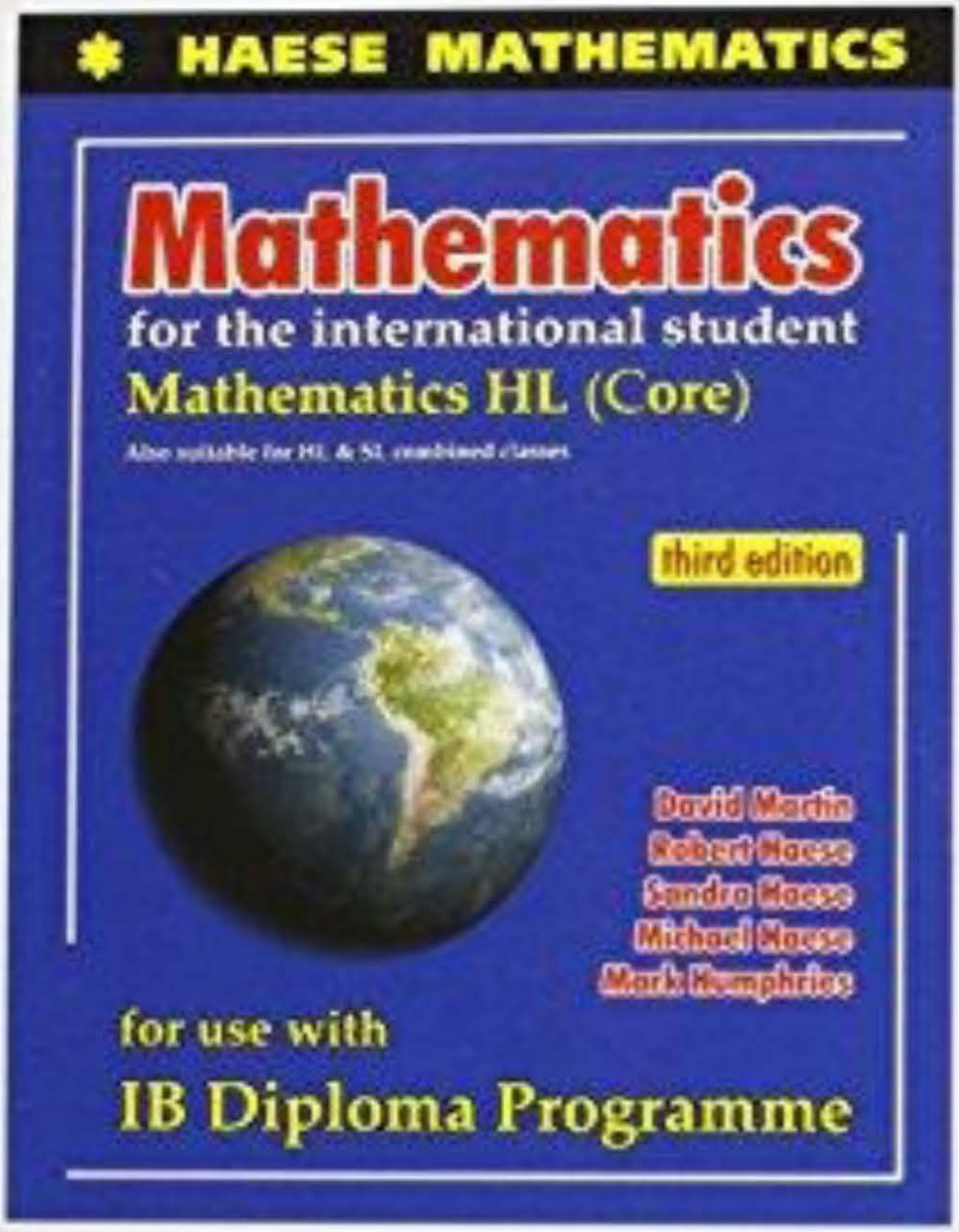
EXAM PREPARATION & PRACTICE GUIDE



HAESE MATHEMATICS

Mathematics

for the international student

Mathematics HL (Core)

Also suitable for HL & SL combined classes

third edition

David Martin
Robert Haese
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for use with

IB Diploma Programme

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Ideal for test & exam preparation

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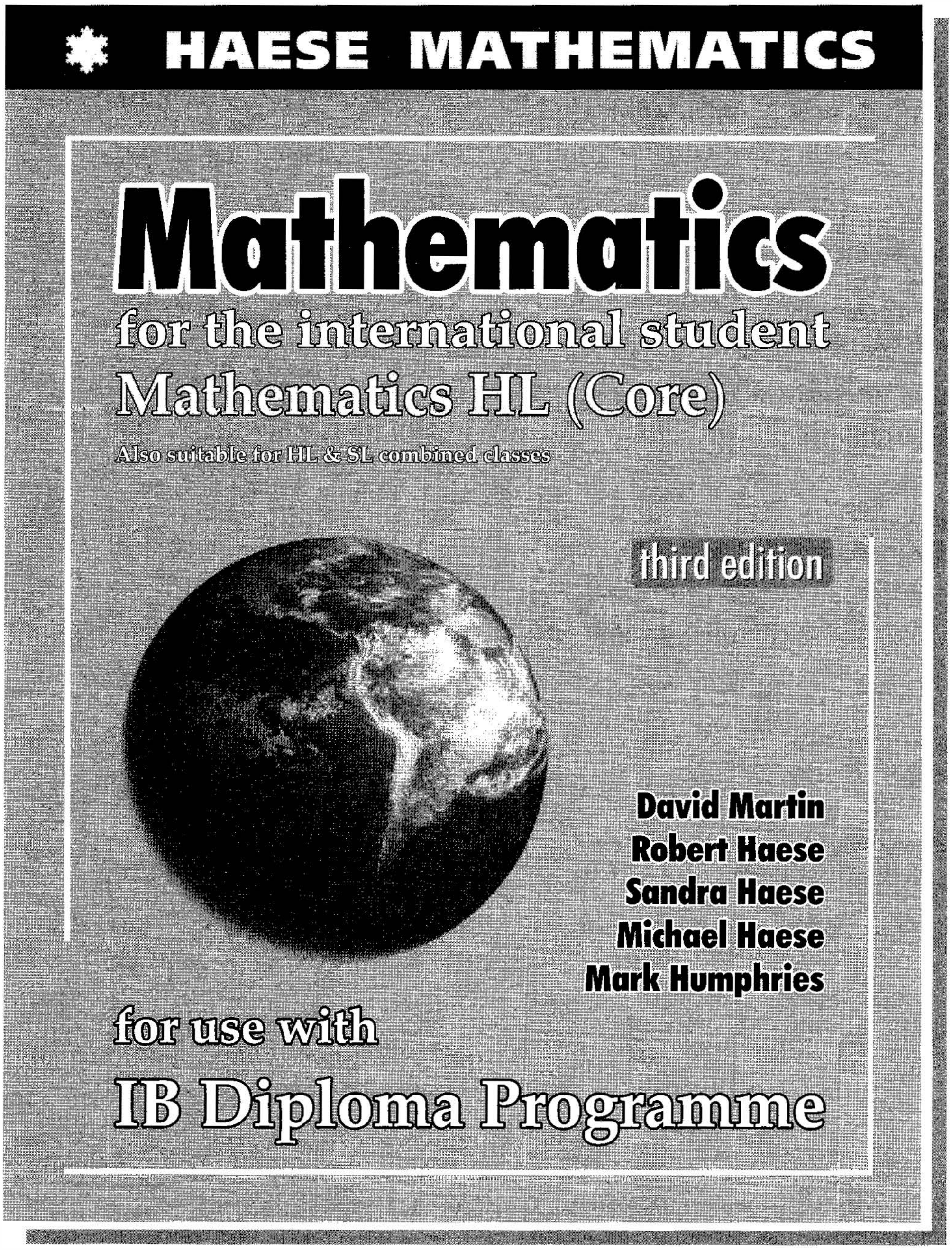

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MATHEMATICS FOR THE INTERNATIONAL STUDENT
Mathematics HL (Core)
EXAM PREPARATION & PRACTICE GUIDE third edition

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FOREWORD

The aim of this guide is to help you prepare for the Mathematics HL Core final examination.

The material covered within this guide is designed to complement your textbook and course information booklet.

The Guide covers all six Topics in the Mathematics HL Core syllabus. Each topic begins with a concise summary highlighting important facts and concepts. Following each summary is a set of Skill Builder Questions; these are designed to remind students of the fundamental skills required for the topic.

There are twelve Exam Sets comprising 25 questions each. Each set is labelled as Calculator or No Calculator questions. These sets can be used as a warm-up; encouraging students to draw on knowledge learnt in class, while helping to identify areas that may need additional practice.

This Guide also offers three Trial Examinations divided into No Calculator and Calculator questions. Detailed marks schemes are provided as a guide for students. This format is consistent with the Mathematics HL Core final examination.

Fully worked solutions are provided for every question in this Guide. It is recommended that you work through a full set of questions or trial exam before checking the solutions.

Try to complete the Trial Examinations under examination conditions. Getting into good habits will reduce pressure during the examination.

- It is important that you persevere with a question, but sometimes it is a good strategy to move on to other questions and return later to ones you have found challenging. Time management is very important during the examination, and too much time spent on a difficult question may mean that you do not leave yourself sufficient time to complete other questions.
- Use a pen rather than a pencil, except for graphs and diagrams.
- If you make a mistake draw a single line through the work you want to replace. Do not cross out work until you have replaced it with something you consider better.
- Set out your work clearly with full explanations. Do not take shortcuts.
- Diagrams and graphs should be sufficiently large, well labelled and clearly drawn.
- Remember to leave answers correct to three significant figures unless an exact answer is more appropriate or a different level of accuracy is requested in the question.

Get used to reading the questions carefully.

- Check for key words. If the word "hence" appears, then you must use the result you have just obtained. "Hence, or otherwise" means that you can use any method you like, although it is likely that the best method uses the previous result.
- Rushing into a question may mean that you miss subtle points. Underlining key words may help.
- Often questions in the examination are set so that, even if you cannot get through one part, the question can still be picked up in a later part.

After completing a practice set, identify areas of weakness.

- Return to your notes or textbook and review the Topic.
- Ask your teacher or a friend for help if further explanation is needed.
- Summarise each Topic. Summaries that you make yourself are the most valuable.
- Test yourself, or work with someone else to help improve your knowledge of a Topic.
- If you have had difficulty with a question, try it again later. Do not just assume that you know how to do it once you have read the solution. It is important that you work on areas of weakness, but do not neglect the other areas.

In addition to the information booklet, your graphics display calculator is an essential aid.

- Make sure you are familiar with the model you will be using.
- In trigonometry questions, remember to check whether the graphics calculator should be in degrees or radians.
- Become familiar with common error messages and how to respond to them.
- Important features of graphs may be revealed by zooming in or out.
- Asymptotic behaviour is not always clear on a graphics calculator screen; don't just rely on appearances. As with all aspects of the graphics calculator, reflect on the reasonableness of the results.
- Are your batteries fresh?

We hope this guide will help you structure your revision program effectively. Remember that good examination techniques will come from good examination preparation.

We welcome your feedback:

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TOPIC 1: ALGEBRA

SEQUENCES AND SERIES

A **number sequence** is an ordered list of numbers defined by a rule. Often, the rule is a formula for the **general term** or **nth term** of the sequence.

A sequence which continues forever is called an **infinite sequence.**

A sequence which terminates is called a **finite sequence**.

ARITHMETIC SEQUENCES

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all n, where d is a constant called the **common difference.**

For an arithmetic sequence with first term u_1 and common difference d, $u_n = u_1 + (n-1)\,d$.

GEOMETRIC SEQUENCES

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant.

$\dfrac{u_{n+1}}{u_n} = r$ for all n, where r is a constant called the **common ratio.**

For a geometric sequence with first term u_1 and common ratio r, $u_n = u_1 r^{n-1}$.

For compound interest problems we have a geometric sequence. If the interest rate is $i\%$ per time period then the common ratio is $\left(1 + \frac{i}{100}\right)$ and the number of compounding periods is n.

SERIES

A **series** is the addition of the terms of a sequence. Given a series which includes the first n terms of a sequence, its sum is

$S_n = u_1 + u_2 + \ldots + u_n = \sum\limits_{k=1}^{n} u_k$.

For an **arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$
or $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

For a **geometric series**, $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$ where $r \neq 1$.

If $|r| < 1$, the sum of an **infinite geometric series** is **convergent**, and $S = \dfrac{u_1}{1-r}$.

If $|r| > 1$ the series is **divergent**.

EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions are inverses of each other. The graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.

Exponent Laws	**Logarithm Laws**
$a^x \times a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \log_a x$
$a^0 = 1 \;\; (a \neq 0)$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$

If $a^x = a^k$ then $x = k$.

To change the base of a logarithm, use the rule $\log_b x = \dfrac{\log_c x}{\log_c b}$.

COUNTING AND THE BINOMIAL THEOREM

THE PRODUCT PRINCIPLE

If there are m different ways of performing an operation and n different ways of performing a second *independent* operation, there are mn ways of performing the two operations in succession.

In counting processes, the word:

- **and** suggests multiplying the possibilities
- **or** suggests adding the possibilities.

FACTORIAL NOTATION

$n! = n\,(n-1)\,(n-2)\ldots \times 3 \times 2 \times 1$ for all $n \geqslant 1$.

$0! = 1$

The **binomial coefficient** is $\dbinom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$

PERMUTATIONS AND COMBINATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

A **combination** is a selection of objects *without regard to order* or arrangement.

The number of combinations on n distinct symbols taken r at a time is $C_r^n = \dbinom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

The **general binomial expansion** is

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots$$
$$+ \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n.$$

MATHEMATICAL INDUCTION

Suppose P_n is a proposition which is defined for every integer $n \geqslant a$, $a \in \mathbb{Z}$.

To construct a formal proof by mathematical induction:

- make sure the proposition is clearly stated
- prove the initial case P_a is true
- prove that if P_k is true then P_{k+1} must also be true
- state your conclusion clearly.

COMPLEX NUMBERS

Any number of the form $a + bi$ where a and b are real and $i = \sqrt{-1}$ is called a **complex number**.

If $z = a + bi$ where a and b are real then:

- a is the **real part** of z, written $\mathcal{R}e(z)$
- b is the **imaginary part** of z, written $\mathcal{I}m(z)$.

Two complex numbers are equal if their real parts are equal *and* their imaginary parts are equal.

$$a + bi = c + di \;\Leftrightarrow\; a = c \text{ and } b = d.$$

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

Properties of complex conjugates:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 - z_2)^* = z_1^* - z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$ and $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*},\ z_2 \neq 0$
- $(z^n)^* = (z^*)^n$ for $n \in \mathbb{Z}^+$
- $z + z^*$ and zz^* are real.

THE COMPLEX PLANE

On the complex plane, $a + bi$ is represented as $\begin{pmatrix} a \\ b \end{pmatrix}$.

The **modulus** of the complex number $z = a + bi$ is the real number $|z| = \sqrt{a^2 + b^2}$.

For points P_1 and P_2 on the complex plane defined by $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$, the distance between P_1 and P_2 is $|z_1 - z_2|$.

If z is represented by the point P(a, b) on the Cartesian plane, and $\overrightarrow{OP}$ makes angle θ with the positive real axis, then θ is called the **argument** of z, or $\arg z$.

Properties of modulus and argument:

- $|wz| = |w|\,|z|$ and $\arg(wz) = \arg w + \arg z$
- $\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|}$ and $\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z$
  provided $|z| \neq 0$
- $|z^*| = |z|$ and $\arg(z^*) = -\arg z$
- $zz^* = |z|^2$

COMPLEX NUMBER REPRESENTATIONS

Cartesian form $z = a + bi$

Polar form $z = |z| \operatorname{cis}\theta$ where $\operatorname{cis}\theta = e^{i\theta} = \cos\theta + i\sin\theta$

$\operatorname{cis} 0 = 1$, $\operatorname{cis}\frac{\pi}{2} = i$, $\operatorname{cis}\pi = -1$, $\operatorname{cis}\frac{3\pi}{2} = -i$

$\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$

$\dfrac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta - \phi)$

$\operatorname{cis}(\theta + k2\pi) = \operatorname{cis}\theta$ for all $k \in \mathbb{Z}$.

DE MOIVRE'S THEOREM

$(|z| \operatorname{cis}\theta)^n = |z|^n \operatorname{cis} n\theta$

The ***n*th roots of the complex number** c are the n solutions of $z^n = c$.

There are *exactly* n nth roots of c.

If $c \in \mathbb{R}$, the roots must occur in conjugate pairs. So, if w is a complex, non-real zero of the real polynomial $P(z)$, then w^* is also a zero, and $z^2 - (w + w^*)z + ww^*$ is a real quadratic factor of $P(z)$.

The nth roots of c all have the same modulus, $|c|^{\frac{1}{n}}$.

The ***n*th roots of unity** are the n solutions of $z^n = 1$.

SYSTEMS OF LINEAR EQUATIONS

You should be able to solve systems of linear equations, up to three equations in three unknowns, using algebra and technology.

Using algebra, we write the system in **augmented matrix form**, then use **elementary row operations** to **reduce** the system to **echelon form**. The three legitimate row operations are to:

- interchange rows
- replace any row by a non-zero multiple of itself
- replace any row by itself, plus or minus a multiple of another row.

We can hence determine whether there are **no solutions** (the system is inconsistent), a **single solution**, or **infinitely many solutions**.

SKILL BUILDER QUESTIONS

1 Find the sum of the first 30 terms of:
  a $18 + 16 + 14 + 12 + ...$
  b $48 + 24 + 12 + 6 + ...$

2 Write $\dfrac{x^a\sqrt{x^{3a}}}{x^{-2a}}$ as a single power of x.

3 Find: **a** $\log_4 8$ **b** $\log_9\left(\frac{1}{27}\right)$ **c** $\log_{\frac{1}{3}}(\sqrt{3})$

4 Expand and simplify: $(2 - ai)^3$

5 Consider the binomial expansion of $\left(2x - \dfrac{1}{x^2}\right)^{12}$. Find:
  a the coefficient of x^3 **b** the constant term.

6 Find the sum of the infinite geometric series with first term 27 and fourth term 8.

7 Write in simplest form: $\left(\dfrac{3x^{-1}}{2a^2}\right)^{-2} \times \left(\dfrac{4x^2}{27a^{-3}}\right)^{-1}$

8 Solve for x: $\log_5(2x - 1) = -1$

9 Write $3 - 3i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$.

10 Illustrate $\{z \mid \mathcal{Re}(z) \leqslant 1 \cap 0 \leqslant \mathcal{Im}(z) \leqslant 3\}$.

11 Prove that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ for positive integers n and r, $r \leqslant n$.

12 A sequence is defined by $u_n = \dfrac{2n+1}{3}$ for $n \in \mathbb{Z}^+$.
  a Prove that the sequence is arithmetic.
  b Find the 50th term and the sum of the first 50 terms.
  c Is 117 a term of the sequence?
  d Find:
    i $\sum\limits_{n=1}^{40} u_n$ **ii** $\sum\limits_{n=30}^{60} u_n$

13 Find x if $8^{2x-3} = 16^{2-x}$.

14 Write $\dfrac{8}{\log_5 9}$ in the form $a\log_3 b$ where $a, b \in \mathbb{Z}$.

15 Simplify $\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)^{10}$, giving your answer in the form $x + iy$ where $x, y \in \mathbb{R}$.

16 In a party game, each person has a card with the 12 numbers from 1 to 12 printed on it. Each person is asked to put a cross through 3 of the numbers, for example 2, 7, and 10. A prize is won if the three numbers crossed are the same as the three numbers chosen by the host.
How many different possible combinations of three numbers are there?

17 Prove that $\sum\limits_{r=0}^{n} \binom{n}{r} = 2^n$ for $n \in \mathbb{Z}^+$.

18 The sum of an infinite geometric series is 1.5, and its first term is 1. Find:

a the common ratio

b the sum of its first 7 terms, in rational form.

19 Simplify: $\dfrac{3^{x+1} - 3^x}{2(3^x) - 3^{x-1}}$

20 Write $2\ln x + \ln(x-1) - \ln(x-2)$ as a single logarithm.

21 Write in the form $a + bi$ where $a, b \in \mathbb{Q}$:

a $\dfrac{3+4i}{1-3i}$ **b** $\dfrac{3}{i}\left(\dfrac{1}{\sqrt{5}} - \dfrac{2i}{\sqrt{5}}\right)^2$

22 The coefficient of x^3 in the expansion of $(3x+2)^n$ is 21 times the coefficient of x. Find n.

23 Find the value(s) of a for which the system of equations

$$\begin{cases} 3x - ay + 2z = 4 \\ x + 2y - 3z = 1 \\ -x - y + z = 12 \end{cases}$$ has a unique solution.

24 Suppose $\dfrac{z+2}{z-2} = i$. Find z in the form $a + bi$ where $a, b \in \mathbb{R}$.

25 Consider all 4-digit integers where all the digits are different and the first digit is non-zero.

a How many of these numbers are there?

b How many of these numbers have a 7 as one of the four digits?

26 Prove that $1 \times 2 + 2 \times 5 + 3 \times 8 + ... + n(3n-1) = n^2(n+1)$ for $n \in \mathbb{Z}^+$.

27 The 7th and 15th terms of an arithmetic sequence are 1 and -23 respectively. Find:

a the 27th term

b the sum of the first 27 terms of the sequence.

28 Solve for x: $4^x + 4 = 17(2^{x-1})$

29 Write as a logarithmic equation in base b:

a $M = ab^3$ **b** $D = \dfrac{a}{b^2}$

30 z is a complex number and z^* is its complex conjugate. Show that if $z^2 = (z^*)^2$ then z is either real or purely imaginary.

31 At a reunion between 6 men and 5 women, each person shakes hands once with every other person. Find:

a the total number of handshakes

b the number of handshakes between a man and a woman.

32 Write $1 - i$ in polar form, and hence find $(1-i)^{11}$ in Cartesian form.

33 A sequence is defined by $u_n = 12\left(\frac{2}{3}\right)^{n-1}$.

a Prove that the sequence is geometric.

b Find the 5th term in rational form.

c Find, correct to 4 decimal places where appropriate:

i $\sum\limits_{n=1}^{\infty} u_n$ **ii** $\sum\limits_{n=1}^{20} u_n$

34 Find a and b given that $2^a 8^b = \frac{1}{2}$ and $\dfrac{3^{-a}}{3^{b+1}} = 9$.

35 **a** Find the roots of $z^5 = 1$ and display them on a fully labelled Argand diagram.

b If the roots found in **a** are $1, w, w^2, w^3$ and w^4 where w is the root with smallest positive argument, show that $1 + w + w^2 + w^3 + w^4 = 0$.

36 Use complex number methods to write $\sin 3\theta$ in the form $a\sin\theta + b\sin^3\theta$.

37 Solve for x: $\log_3 x + \log_3(x-2) = 1$

38 By considering $(1+x)^{2n} = (1+x)^n(1+x)^n$ show that for $n \in \mathbb{Z}^+$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2.$$

39 Solve for z: $z^2 - z + 1 + i = 0$

40 Prove that $5n^3 - 3n^2 - 2n$ is divisible by 6 for $n \in \mathbb{Z}^+$.

41 A sequence has consecutive terms $k+1$, $3k$, and k^2+5, where $0 < k < 5$.
Find k if the sequence is:

a arithmetic **b** geometric.

42 Prove that $(z+w)^* = z^* + w^*$.

43 **a** Write $z = \dfrac{1 + i\sqrt{3}}{1+i}$ in the form $r \operatorname{cis}\theta$, $r > 0$.

b Hence, find the smallest positive value of n such that

i $z^n \in \mathbb{R}$ **ii** z^n is purely imaginary.

44 Find k given that the constant term of

$$\left(kx + \frac{1}{\sqrt{x}}\right)^9 \text{ is } -10\tfrac{1}{2}.$$

45 **a** Perform row reduction on the system of equations $\begin{cases} x + 3y + kz = 2 \\ kx - 2y + 3z = k \\ 4x - 3y + 10z = 5. \end{cases}$

b Show that for one value of k, the system of equations has infinitely many solutions.

c Find the value(s) of k for which the system has no solutions.

d Find the value(s) of k for which the system has a unique solution.

46 If $\log_a 5 = x$, find in terms of x:

a $\log_a(5a)$ **b** $\log_a\left(\dfrac{a^2}{25}\right)$.

47 u_n is a geometric sequence in which $u_3 = 20$ and $u_6 = 160$. Find:

a u_1 and the common ratio

b u_{10} and $\sum\limits_{n=1}^{12} u_n$.

48 48 people are about to get on a double-decker bus which seats 24 people on each level. However, 8 people refuse to travel upstairs, and 6 refuse to travel downstairs. How many ways are there of choosing which passengers travel upstairs and which passengers travel downstairs?

49 **a** Find the integer b such that $0 < b \leqslant 9$, and $9^n + b$ is divisible by 8 for $n \in \mathbb{Z}^+$.

b Prove your answer in **a** using the principle of mathematical induction.

50 Prove that $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

51 Three numbers are consecutive terms of an arithmetic sequence. Their sum is 18, and the sum of their squares is 396. Find the numbers.

52 Write without logarithms:

a $\log_{10} M = 2x - 1$

b $\log_a N = 2\log_a d - \log_a c$

53 For what values of k does $\dfrac{z^2+3}{z^2-1} = k$ have imaginary roots?

54 Find the coefficient of x^5 in the expansion of $(x+2)(1-x)^{10}$.

55 Write $z = \dfrac{-1+5i}{2+3i}$ in polar form, and hence show that $z^{12} = -64$.

56 The solution of $2^{x-1} = 3^{2-x}$ is $x = \log_a b$ where $a, b \in \mathbb{Z}^+$. Find a and b.

57 The first term of a finite arithmetic series is 18, the sum of the series is -210, and the common difference is -3. Find the number of terms in the series.

58 Suppose $z = x + iy$ where $x, y \in \mathbb{R}$. If $|z-3| = |z-1|$, deduce that $x = 2$.

59 Of 11 given points, 4 lie on a straight line and no other three points are collinear. How many different straight lines can be drawn through pairs of given points?

60 Prove that $1 + 2\times 2^1 + 3\times 2^2 + ... + n\times 2^{n-1} = (n-1)2^n + 1$ for $n \in \mathbb{Z}^+$.

61 Consider the system of equations $\begin{cases} x - 2y + 3z = 4 \\ 2x - 3y + 2z = 1 \\ 3x - 4y + kz = -2 \end{cases}$

where k is a constant.

a Show that there is a unique solution provided that k does not take one specific value k_1. Find k_1.

b Find the unique solution.

c Discuss the case $k = k_1$.

62 Suppose $z_1 = 3 + 4i$ and $z_2 = 8 - 8i$.

a Find $|z_1|$, $\arg z_1$, $|z_2|$, and $\arg z_2$.

b Show the points $\text{P}(z_1)$, $\text{Q}(z_2)$, and $\text{R}(z_1 + z_2)$ on the complex plane, and explain how point R is located geometrically from P and Q.

63 Deduce that $n\binom{n-1}{r-1} = r\binom{n}{r}$ for $n,\ r \in \mathbb{Z}^+,\ n \geqslant r$.

64 Suppose z and w are complex numbers where $w \neq 0$. Use polar coordinates to show that $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$.

65 **a** Show that $4x^2 \geqslant (x+1)^2$ for $x \geqslant 1$.

b Use the principle of mathematical induction to prove that $4^n \geqslant 3n^2$ for $n \in \mathbb{Z}^+$.

66 Suppose $z = r \operatorname{cis}\theta$ where $r > 0$ and $z^2 = z^*$.

a Deduce that $r^2 = r$ and $\operatorname{cis} 3\theta = 1$.

b Hence show that $z^2 = z^*$ has three non-zero solutions, and write them in the form $a + bi$ where $a, b \in \mathbb{R}$.

67 Stan invests £3500 for 33 months at an interest rate of 8% p.a. compounded quarterly. What will be its maturing value?

68 A club has 12 members. How many different committees consisting of at least two members can be formed?

69 Suppose $z = iz^*$ where $z = x + iy$ and $x, y \in \mathbb{R}$. Deduce that $x = y$.

70 Deduce that $n^3 + 2n$ is divisible by 3 for $n \in \mathbb{Z}^+$.

71 Find the sum of all integers between 100 and 200 (inclusive) which are **not** divisible by 4.

72 Given that $\operatorname{cis}\theta = e^{i\theta}$, deduce that:

a $\operatorname{cis}\theta \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$ **b** $(r \operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$

c if $w = e^{i\left(\frac{2\pi}{5}\right)}$, then $(1+w)(1+w^2) = -w^4$.

73 Find b given that the coefficient of x^{-3} in

$$\left(\sqrt{x} + \frac{b}{x}\right)^9 \text{ is } -4032.$$

74 Find:

a n if $\sum\limits_{k=1}^{n}(3k - 11) = 5536$

b y if $\sum\limits_{k=1}^{\infty}\left(\dfrac{y}{5}\right)^{k-1} = 5$.

75 A teacher needs to decide the order in which to schedule 8 examinations, two of which are Mathematics A and Mathematics B. In how many ways can this be done given that the two Mathematics subjects must not be consecutive?

76 Suppose $w = e^{i\left(\frac{2\pi}{3}\right)}$.

a Deduce that $w^3 = 1$ and $1 + w + w^2 = 0$.

b Write in terms of w, in simplest form:

i w^7 **ii** w^{-1} **iii** $(1-w)^2$

iv $\dfrac{1}{(1+w)^2}$ **v** $\dfrac{1+w^2}{1+w}$

77 Use the principle of mathematical induction to prove that $3^n > n^2 + n$ for $n \in \mathbb{Z}^+$.

78 **a** Find the cube roots of $-27i$ and display them on an Argand diagram, labelling them z_1, z_2, and z_3.

b Show that $z_2 z_3 = z_1^2$, where z_1 is any of the cube roots found in **a**.

c What is the value of $z_1 z_2 z_3$?

79 5 distinct points lie on a circle and 11 distinct points lie within it. No three points are collinear. How many different triangles can be drawn with vertices selected from the 16 points if:

a there are no other restrictions

b exactly one of the vertices lies on the circle

c at least one of the vertices lies within the circle.

80 **a** If $z = \operatorname{cis}\theta$, prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ for $n \in \mathbb{Z}^+$

b By considering $\left(z + \dfrac{1}{z}\right)^4$, write $\cos^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta + c$ where $a, b, c \in \mathbb{Q}$.

81 The points $(2, 4)$, $(2, -6)$, and $(-1, 3)$ lie on a circle with equation $x^2 + y^2 + ax + by + c = 0$.

a Write three equations in the unknowns a, b, and c.

b Find the values of a, b, and c, and hence find the coordinates of the circle's centre.

82 Emma sets up a fund for her granddaughter, Amy. On the first day of each month, Emma deposits \$60 in an account. The account pays compound interest of 5% per annum, calculated monthly. The interest is added to the account on the last day of each month.

a Find the value of the fund after 3 months.

b Write an expression for the value of the fund after k years.

c Hence find the value of the fund after 20 years.

83 In a geometric sequence, the first term is u_1 and the nth term is u_n.

a Find an expression for the sum of the reciprocals of the first n terms.

b Hence find $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+....+\frac{1}{3072}$

TOPIC 2: FUNCTIONS AND EQUATIONS

FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** is any set of points which connect two variables.

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first member. So, for each value of x there is only one value of y or $f(x)$.

We **test for functions** using the vertical line test. A graph is a function if no vertical line intersects the graph more than once. For example, a circle such as $x^2+y^2=1$ has a graph which is not a function.

The **domain** of a relation is the set of values of x in the relation.

The **range** of a relation is the set of values of y in the relation.

To fully describe a function, we need a **rule** *and* a **domain**.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

A function $f(x)$ is **even** if $f(-x)=f(x)$ for all x in the domain of f.

A function $f(x)$ is **odd** if $f(-x)=-f(x)$ for all x in the domain of f.

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g is $f \circ g : x \mapsto f(g(x))$.

In general, $f(g(x)) \neq g(f(x))$, so $f \circ g \neq g \circ f$.

The **identity function** is $f(x)=x$.

A relation is:

- **one-to-one** if there is only one y for each x *and* only one x for each y.
- **many-to-one** if there is more than one value of x with the same value of y.

The function $y=f(x)$ has an **inverse function** $y=f^{-1}(x)$ if and only if it is one-to-one.
A many-to-one function will not have an inverse.

If $y=f(x)$ has an inverse function, then the inverse function:

- is a reflection of $y=f(x)$ in the line $y=x$
- satisfies $(f \circ f^{-1})(x)=(f^{-1} \circ f)(x)=x$
- has range equal to the domain of $f(x)$
- has domain equal to the range of $f(x)$.

A **reciprocal function** has the form $f(x)=\dfrac{k}{x}$, where $k \neq 0$ is a constant.
The reciprocal function is a **self-inverse function**, as $f^{-1}(x)=f(x)=\dfrac{k}{x}$.

GRAPHS OF FUNCTIONS

The **x-intercepts** of a function are the values of x for which $y=0$. They are the **zeros** of the function.

The **y-intercept** of a function is the value of y when $x=0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.

To find vertical asymptotes, look for values of x for which the function is undefined:

- if $y=\dfrac{f(x)}{g(x)}$ find where $g(x)=0$
- if $y=\log_a(f(x))$ find where $f(x)=0$.

To find horizontal asymptotes, consider the behaviour as $x \to \pm\infty$.

TRANSFORMATIONS OF GRAPHS

- $y=f(x)+b$ **translates** $y=f(x)$ vertically b units.
- $y=f(x-a)$ **translates** $y=f(x)$ horizontally a units.
- $y=f(x-a)+b$ **translates** $y=f(x)$ by the vector $\binom{a}{b}$.
- $y=pf(x), p>0$ is a **vertical stretch** of $y=f(x)$ with scale factor p.
- $y=f(qx), q>0$ is a **horizontal stretch** of $y=f(x)$ with scale factor $\dfrac{1}{q}$.
- $y=-f(x)$ is a **reflection** of $y=f(x)$ in the x-axis.
- $y=f(-x)$ is a **reflection** of $y=f(x)$ in the y-axis.
- If $f^{-1}(x)$ exists, $y=f^{-1}(x)$ is a **reflection** of $y=f(x)$ in the line $y=x$.

Invariant points are the points which do not move under a transformation.

MODULUS FUNCTIONS

The **absolute value** or **modulus** function

$$f(x)=|x|=\begin{cases} x, & \text{if } x \geqslant 0 \\ -x, & \text{if } x<0. \end{cases}$$

Properties of modulus for all x and y:

- $|x| \geqslant 0$
- $|x|^2=x^2$
- $\left|\dfrac{x}{y}\right|=\dfrac{|x|}{|y|}$
- $|-x|=|x|$
- $|xy|=|x|\,|y|$
- $|x-y|=|y-x|$

To graph $y=|f(x)|$, we reflect in the x-axis that part of the graph of $y=f(x)$ which lies below the x-axis; the rest remains invariant.

To graph $y=f(|x|)$, the part of the graph of $y=f(x)$ which lies to the right of the y-axis is invariant, and this part is reflected in the y-axis.

GRAPHS OF RECIPROCAL FUNCTIONS

- If $f(x) > 0$ then $\dfrac{1}{f(x)} > 0$.
- If $f(x) < 0$ then $\dfrac{1}{f(x)} < 0$.
- When $f(x)$ is a minimum, $\dfrac{1}{f(x)}$ is a maximum, and vice versa.
- Zeros of $f(x)$ correspond to vertical asymptotes of $\dfrac{1}{f(x)}$.
- Vertical asymptotes of $f(x)$ correspond to zeros of $\dfrac{1}{f(x)}$.
- Invariant points occur when $f(x) = \pm 1$.

RATIONAL FUNCTIONS

You should be able to graph a rational function of the form $y = \dfrac{ax+b}{cx+d}$, $c \neq 0$ using transformations, and include its asymptotes.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The simplest **exponential function** is $f(x) = a^x$, $a > 0$, $a \neq 1$.

If $a > 1$ we have *growth*. If $0 < a < 1$ we have *decay*.

The graph of $y = a^x$ has the horizontal asymptote $y = 0$.

The inverse function of $f(x) = a^x$ is the **logarithmic function** $f^{-1}(x) = \log_a x$, $x > 0$.

The graph of $y = \log_a x$ has the vertical asymptote $x = 0$.

The equation $a^x = b$ may in some cases be solved by writing b as a power with base a. If this cannot be done, use logarithms.

The exponential with base e corresponds to the **natural logarithm** ln.

QUADRATICS

A **quadratic function** has the form $y = ax^2 + bx + c$, $a \neq 0$.

A **quadratic equation** has the form $ax^2 + bx + c = 0$, $a \neq 0$. It can be solved by:

- factorisation
- completing the square
- the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The **discriminant** of the quadratic equation is $\Delta = b^2 - 4ac$.

The quadratic equation has:

- *no real solutions* if $\Delta < 0$
- *one real (repeated) solution* if $\Delta = 0$
- *two real solutions* if $\Delta > 0$.

The number of solutions indicates whether the graph of the corresponding quadratic function *does not meet*, *touches*, or *cuts* the x-axis.

If $ax^2 + bx + c = 0$ has roots α and β, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

The graph is a **parabola** with the following properties:

- it is *concave upwards* if $a > 0$ and *concave downwards* if $a < 0$
- its axis of symmetry is $x = \dfrac{-b}{2a}$
- its vertex is at $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$
- its y-intercept is c.

A quadratic function written in the form:

- $x \mapsto a(x-h)^2 + k$ has vertex (h, k)
- $x \mapsto a(x-p)(x-q)$ has x-intercepts p and q.

The quadratic $y = ax^2 + bx + c$ is:

- positive definite if $a > 0$ and $\Delta < 0$
- negative definite if $a < 0$ and $\Delta < 0$.

REAL POLYNOMIALS

A **real polynomial** is a function of the form $P(x) = \sum\limits_{r=0}^{n} a_r x^r$ where $a_r \in \mathbb{R}$ for all $r = 0, 1, 2, ..., n$.

If $P(x)$ is divided by $ax + b$ until a constant remainder R is obtained:

$$P(x) = Q(x)(ax+b) + R$$

$$\text{where } \begin{cases} ax+b & \text{is the } \textbf{divisor} \\ Q(x) & \text{is the } \textbf{quotient} \\ R & \text{is the } \textbf{remainder}. \end{cases}$$

α is a **zero** of the polynomial $P(x) \Leftrightarrow P(\alpha) = 0$.

α is a **root** of the polynomial equation

$$P(x) = 0 \Leftrightarrow P(\alpha) = 0.$$

$(x - \alpha)$ is a **factor** of the polynomial $P(x)$ if and only if there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

THE REMAINDER THEOREM

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained then $R = P(k)$.

THE FACTOR THEOREM

k is zero of $P(x) \Leftrightarrow (x - k)$ is a factor of $P(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

- Every polynomial of degree $n \geqslant 1$ has at least one zero which can be written in the form $a + bi$ where $a, b \in \mathbb{R}$.
- If $P(x)$ is a polynomial of degree n, then $P(x)$ has exactly n zeros, some of which may either be irrational numbers or complex numbers.

The Fundamental Theorem of Algebra gives the following properties of real polynomials:

- Every **real** polynomial of degree n can be factorised into n complex linear factors, some of which may be repeated.
- Every real polynomial can be expressed as a product of **real** linear and **real** irreducible quadratic factors (where $\Delta < 0$).
- Every **real** polynomial of degree n has exactly n zeros, some of which may be repeated.
- If $p + qi$ $(q \neq 0)$ is a zero of a **real** polynomial then its complex conjugate $p - qi$ is also a zero.
- Every **real** polynomial of odd degree has at least one real zero.

GRAPHS OF POLYNOMIALS

A single zero of a real polynomial indicates its graph *cuts* the x-axis at this point.

A double repeated zero indicates the graph *touches* the x-axis.
A triple repeated zero indicates the graph has a *stationary point of inflection* on the x-axis.

SUM AND PRODUCT OF ROOTS THEOREM

For the polynomial equation $\sum_{r=0}^{n} a_r x^r = 0$, the sum of the roots is $\frac{-a_{n-1}}{a_n}$, and the product of the roots is $\frac{(-1)^n a_0}{a_n}$.

INEQUALITIES IN ONE VARIABLE

You should be able to:

- solve inequalities graphically
- use the absolute value sign in inequalities
- solve $g(x) \geqslant f(x)$ algebraically where f and g are linear, quadratic, or simple cubic functions
- use sign diagrams.

Rules for manipulating inequalities:

- $a > b$ and $c \in \mathbb{R} \Rightarrow a + c > b + c$
- $a < b$ and $c \in \mathbb{R} \Rightarrow a + c < b + c$
- $a > b$ and $c > 0 \Rightarrow ac > bc$
- $a > b$ and $c < 0 \Rightarrow ac < bc$
- $a > b \geqslant 0 \Rightarrow a^2 > b^2$
- $a < b \leqslant 0 \Rightarrow a^2 > b^2$

SKILL BUILDER QUESTIONS

1 Define the term *function* and state a test for determining whether the graph of a relation is a function.

2 State the domain and range of:

a $g : x \mapsto 4 - \ln(x - 2)$ **b** $f(x) = \frac{1}{x-1} - \sqrt{x+1}$

3 Functions f and g are defined by
$f : x \mapsto 3x + 1$ and $g : x \mapsto 4 - x$.
Find: **a** $f(g(x))$ **b** $(g \circ f)(-4)$ **c** $f^{-1}\left(\frac{1}{2}\right)$

4 Find the equation of the resulting image when $y = \frac{2}{x}$ is:

a reflected in the y-axis

b translated by $\binom{-1}{2}$

c stretched horizontally with scale factor 3.

5 Graph $y = -1 + 2^{-x}$ and include on your graph any axes intercepts and asymptotes.

6 Consider the function $f : x \mapsto 2x^2 + 4x - 20$.

a State the equation of the axis of symmetry.

b State the coordinates of the turning point.

c Write the function in the form:
i $y = a(x - h)^2 + k$ **ii** $y = a(x - p)(x - q)$.

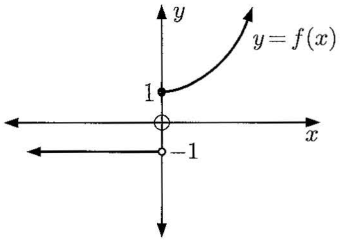
7 **a** Graph $f : x \mapsto e^{x-1}$.

b State the domain and range of f.

8 **a** Graph $f : x \mapsto \ln(x - 2)$.

b State the domain and range of f, and the equation of the asymptote.

9



The function $y = f(x)$ is illustrated. On the same set of axes, sketch graphs of:

a $y = f(x)$ **b** $y = \frac{1}{f(x)}$ **c** $y = f(-x)$

d $y = f(x - 2)$ **e** $y = 2f(x)$ **f** $y = -f(x)$

10 Find $f^{-1}(x)$ where $f(x) = x^2 + 2x$ and $x \in\ ]-\infty, -1]$.
Find any invariant points.

11 Solve for x using algebraic methods:

a $\frac{2x}{x-1} \leqslant \frac{1}{3}$ **b** $x \geqslant \frac{4}{x}$ **c** $\frac{1}{x+2} > \frac{2}{x}$

12 $3x^2 + x - 1 = 0$ has roots p and q. Find $p^2 + q^2$.

13 Consider $f(x) = 2^{x-1}$.

a Graph $y = f(x)$ and its inverse function on the same set of axes.

b Find $f^{-1}(x)$.

14 If $f(x) = x^2 + 2$ and $h(x) = 3 - 2x$, find $(h \circ f)(x)$.

15 Find the remainder when $2x^5 - x^3 + 4x - 1$ is divided by $x + 1$.

16 The line with equation $y = kx - 2$ is a tangent to the quadratic $y = 3x^2 + x + 1$. Find k.

17 **a** Sketch the graph of $y = |x - 2| + |x|$.

b Hence, solve for x:
i $|x - 2| + |x| = 2$ **ii** $|x - 2| + |x| \geqslant 3$

18 Find the inverse function of $f : x \mapsto e^{2x+1}$ and hence find $f^{-1}(7)$.

19 **a** Sketch the graph of $f(x) = x^2 - 2x$, $x \in \mathbb{R}$, showing clearly the x-intercepts and vertex.

b Hence sketch the graphs of:
i $y = f(|x|)$ **ii** $y = |f(x)|$

20 Find d if $x + 2$ is a factor of $2x^3 + 4x^2 + dx - 6$.

21 Find all quadratic equations with roots:

a 0 and $\frac{1}{2}$ **b** $\frac{2}{3}$ and $-\frac{1}{4}$

c $-2 \pm \sqrt{2}$ **d** $-1 \pm i\sqrt{3}$

22 When $P(x) = 2x^n - 10x - 5$ is divided by $x + 2$ the remainder is 47. Find n.

23 Write $9x^4 + 4$ as a product of quadratic factors.

24 If $g(x) = \log_2(2x - 1)$, find $g^{-1}(x)$ and hence evaluate $g^{-1}(-6)$.

25 Find m given that $mx^2 + (m - 2)x + m = 0$ has a repeated root.

26 **a** Determine whether the following functions are odd, even, or neither:

i $y = x - \frac{1}{x}$ **ii** $y = \cos 2x$

b **i** Can an even function have an inverse? Explain your answer.

ii What domain restriction could be placed on $y = x^4 + x^2$ so that the new function obtained has an inverse?

27 -2 is a solution of $x^2 + bx + (b-2) = 0$. Find b and the other solution.

28 Write $\dfrac{2x^3 - 11x^2 + 21x - 8}{(x-2)^2}$ in the form $ax + b + \dfrac{c}{x-2} + \dfrac{d}{(x-2)^2}$.

29 Sketch the function $f(x) = 3(x-1)^2(x+2)(x-4)$, showing clearly the axes intercepts.

30 **a** Sketch the graph of $f : x \mapsto \dfrac{2x+1}{x-2}$, clearly showing any axes intercepts and asymptotes.

b Hence, sketch the graph of $y = \dfrac{1}{f(x)}$ on the same set of axes.

c State the position of any invariant points.

31 Find $a \in \mathbb{R}$ given that $1+i$ is a zero of $x^3 + ax^2 - 4ax + 6$.

32 Find the quotient and remainder when $x^4 + 3x^2 + x$ is divided by:

a $x+2$ **b** $(x-1)^2$.

33 On the same axes, sketch the graphs of $y = e^{x-2}$ and $y = 2 - e^x$ for $x \in [-1, 5]$.
Hence find, correct to 3 decimal places, any points of intersection of the graphs.

34 $y = 3x^2 + 2x$ is stretched vertically by a factor of 2 and then translated by $\begin{pmatrix}3\\-1\end{pmatrix}$. Find the equation of the resulting image.

35 $x^2 + mx + 1 = 0$ has roots α and β. Find m if $\alpha\beta = \dfrac{1}{\alpha} + \dfrac{1}{\beta}$.

36 Find the values of d such that $y = dx + 2$ meets $y = x^2 + 3x + 3$ in two distinct points.

37 $x-1$ and $x+3$ are factors of $P(x) = 2x^3 + ax^2 + bx - 3$. Find a and b and the three zeros of $P(x)$.

38 Sketch the graph of:

a $y = |3-x|$ **b** $y = |3-x| - |x|$

39 $a + 2i$ is a root of $z^2 + bz + (a+6) = 0$. Find a and b given that $a, b \in \mathbb{R}$.

40 State the asymptotes of the graph with equation:

a $y = 4 - \ln(x-2)$ **b** $y = \dfrac{3+x}{2x-1}$

c $y = 2e^{x-4}$

41 Factorise $x^3 + bx^2 + ax + ab$ into linear factors.

42 A quartic polynomial $P(x)$ has a graph which cuts the y-axis at 56, cuts the x-axis at -1, touches the x-axis at 2 and passes through the point $(1, 20)$.
Find $P(x)$ in expanded form.

43 Functions f and g are given by $f : x \mapsto e^{x+1}$ and $g : x \mapsto \ln x - 1$.

a Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

b Graph $y = f(x)$ and $y = g(x)$ on the same set of axes.

c State the relationship between f and g.

44 Given that the graph of $f(x) = mx^2 + (m-1)x + 2$ does not cut the x-axis, find the possible values of m.

45 When the polynomial $P(x)$ is divided by $(x-1)(x-2)$, the remainder is $2x+3$.
What is the remainder when $P(x)$ is divided by $x-1$?

46 Solve for x: $|x-1| \leqslant \dfrac{x}{2}$.

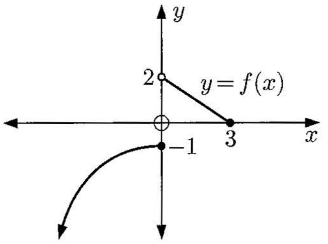
47 Find m given that $x^3 + mx + m$ leaves a remainder of m when divided by $x - m$.

48 $(x-1)^2$ is a factor of $P(x) = x^4 + ax^3 + 2x^2 + bx - 3$.

a Find a and b.

b Sketch the graph of $y = P(x)$.

49



The function $y = f(x)$ is illustrated.

a State the domain and range of $y = f(x)$.

b On the same set of axes, sketch graphs of:

i $y = -2f(x)$ **ii** $y = |f(x)|$

c Suppose $g(x) = f(|x|)$.

i Is $g(x)$ odd or even?

ii Does $g(x)$ have an inverse?

50 $x = 2$ is a zero of $x^3 - x^2 + (m+1)x + (2 - m^2)$.
Find m and hence show that no other real zeros exist.

51 Suppose $P(x) = 2x^m + 3x^n + p$ where $m, n \in \mathbb{Z}^+$, $p \in \mathbb{Z}$. $P(x)$ leaves a remainder of 6 when divided by $x-1$, and a remainder of 77 when divided by $x-2$. Find m, n, and p.

52 Consider $P(z) = 6z^4 + 7z^3 + 8z^2 + 7z + k$. Given that i is a zero of $P(z)$, find k and the remaining zeros of $P(z)$.

53 If $a > b > 0$, prove that $\dfrac{1}{a} < \dfrac{1}{b}$.

54 Given the function $f : x \mapsto \sqrt[3]{x}$, find an expression for $g(x)$ in terms of x in each of the following cases.

a $(f \circ g)(x) = 2x - 1$ **b** $(g \circ f)(x) = 2x - 1$

55 The population P of a species after n months follows the rule $P = 1000 + ae^{kn}$.
Given that initially the population was 2000 and after 1 year the population was 4000, find how long it will take for the population to reach 10 000.

56 $x = a$ is a solution of $3x^3 - 11x^2 + 8x = 12a$.

a Show that there are 3 possible values for a.

b For each value of a found in **a**, solve the original equation.

57 $3 - 2i$ is a zero of $P(x) = 2x^3 + mx^2 - (m+1)x + (3-4m)$, $m \in \mathbb{R}$. Find m and the other two zeros of $P(x)$.

58 $z = a$ is a zero of $P(z) = a^2z^3 + z^2 - a^4z - 2$. Find a and the other zeros of $P(z)$.

59 Given the functions $f : x \mapsto 2x - 1$ and $g : x \mapsto 2x^3$, find the function $(f \circ g)^{-1}$.

60 One zero of $x^4 + 2x^3 + 8x^2 + 6x + 15$ has form bi where $b \neq 0$, $b \in \mathbb{R}$. Find b and all zeros of the polynomial.

TOPIC 3: CIRCULAR FUNCTIONS AND TRIGONOMETRY

RADIAN MEASURE

There are $360° \equiv 2\pi$ radians in a circle.

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

APPLICATIONS OF RADIANS

For θ in radians:

- the length of an arc of radius r and angle θ is $l = r\theta$
- the area of a sector of radius r and angle θ is $A = \frac{1}{2}\theta r^2$
- the area of a segment of radius r and angle θ is $A = \frac{1}{2}\theta r^2 - \frac{1}{2}r^2 \sin\theta$.

THE UNIT CIRCLE

The **unit circle** is the circle centred at the origin O, with radius 1 unit.

The coordinates of any point P on the unit circle, where the angle θ is made by [OP] and the positive x-axis, are $(\cos\theta, \sin\theta)$.

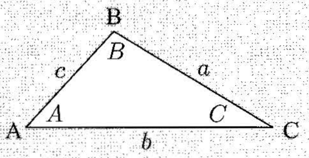
θ is **positive** when measured in an **anticlockwise** direction from the positive x-axis.

$\tan\theta$ is defined as $\dfrac{\sin\theta}{\cos\theta}$.

You should memorise or be able to quickly find the values of $\cos\theta$, $\sin\theta$, and $\tan\theta$ for θ that are multiples of $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

For the triangle alongside:



Area formula $\text{Area} = \frac{1}{2}ab\sin C$

Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

Sine rule $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

If you have the choice of rules to use, use the cosine rule to avoid the **ambiguous case**.

THE GENERAL SINE FUNCTION

If we begin with $y = \sin x$, we can perform transformations to produce the **general sine function** $f(x) = a\sin(b(x - c)) + d$.

We have a vertical stretch with factor $|a|$, a reflection in the x-axis if $a < 0$, then a horizontal stretch with factor $\frac{1}{b}$, and finally a translation with vector $\binom{c}{d}$.

For the general sine function:

- the **amplitude** is $|a|$
- the **principal axis** is $y = d$
- the **period** is $\frac{2\pi}{b}$.

OTHER TRIGONOMETRIC FUNCTIONS

You should be able to graph and use:

- the cosine function $y = a\cos(b(x - c)) + d$
- the tangent function $y = a\tan(b(x - c)) + d$

$y = \tan bx$ has period $\frac{\pi}{b}$.

RECIPROCAL TRIGONOMETRIC FUNCTIONS

$\operatorname{cosec} x$ or $\csc x = \dfrac{1}{\sin x}$.

$\operatorname{secant} x$ or $\sec x = \dfrac{1}{\cos x}$.

$\operatorname{cotangent} x$ or $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$.

When graphing $\csc x$, $\sec x$, and $\cot x$, there will be vertical asymptotes corresponding to the zeros of $\sin x$, $\cos x$, and $\tan x$. $\cot x$ will have zeros corresponding to the vertical asymptotes of $\tan x$.

TRIGONOMETRIC IDENTITIES

$\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for all $k \in \mathbb{Z}$

NEGATIVE ANGLES

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$

COMPLEMENTARY ANGLES

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

PYTHAGOREAN IDENTITIES

$\cos^2\theta + \sin^2\theta = 1$, $\tan^2 x + 1 = \sec^2 x$, and $1 + \cot^2 x = \csc^2 x$

DOUBLE ANGLE FORMULAE

$\sin 2A = 2\sin A\cos A$

$$\cos 2A = \begin{cases} \cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1 \end{cases}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

COMPOUND ANGLE FORMULAE

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

TRIGONOMETRIC EQUATIONS

To solve trigonometric equations we can either use graphs from technology, or algebraic methods involving the trigonometric identities. In either case we must make sure to include all solutions on the specified domain.

We need to use the inverse trigonometric functions to invert sin, cos, and tan.

Function	*Domain*	*Range*
$x \mapsto \arcsin x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$x \mapsto \arccos x$	$[-1, 1]$	$[0, \pi]$
$x \mapsto \arctan x$	$]-\infty, \infty[$	$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

The ranges of these functions are important because our calculator will only give us the one answer in the range. Remember that other solutions may also be possible. For example, when using arcsin our calculator will always give us an acute angle answer, but the obtuse angle with the same sine may also be valid.

An equation of the form $a\sin x = b\cos x$ can always be solved as $\tan x = \frac{b}{a}$.

SKILL BUILDER QUESTIONS

1 Convert:

a $\frac{2\pi}{9}$ radians to degrees **b** $140°$ to radians.

2 Find the exact value of:

a $\sin\left(\frac{5\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(-\frac{\pi}{3}\right)$

3 A sector of a circle of radius 10 cm has a perimeter of 40 cm. Find the area of the sector.

4 What consecutive transformations map the graph of $y = \sin x$ onto:

a $y = 2\sin\left(\dfrac{x}{3}\right)$ **b** $y = \sin\left(x + \frac{\pi}{3}\right) - 4$?

5 Find the amplitude, principal axis, and period of the following functions:

a $f(x) = \sin 4x$ **b** $f(x) = -2\sin\left(\dfrac{x}{2}\right) - 1.$

6 Sketch the graph of $y = \csc(x)$ for $x \in [0,\ 3\pi]$.

7 Simplify $\sin\left(\frac{3\pi}{2} - \phi\right)\tan(\phi + \pi)$.

8 If $\cos(2x) = \frac{5}{8}$, find the exact value of $\sin x$.

9 Solve $\sin 2x = \sin x$ for $x \in [-\pi,\ \pi]$, giving exact answers.

10 A sector of a circle has an arc length of 6 cm and an area of 20 cm^2. Find the angle of the sector.

11 Find the period of:

a $y = -\sin(3x)$ **b** $y = 2\sin\left(\dfrac{x}{2}\right) + 1$

c $y = \sin^2 x + 5.$

12 Sketch the graph of $y = \arccos x$, clearly showing the axes intercepts and endpoints.

13 Simplify $1 - \dfrac{\sin^2\theta}{1 + \cos\theta}$.

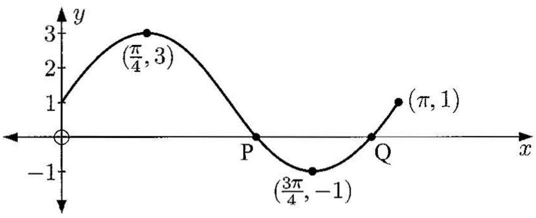
14 If $\tan\theta = 2$, find the exact values of $\tan 2\theta$ and $\tan 3\theta$.

15 Show that $\csc(2x) - \cot(2x) = \tan x$ and hence find the exact value of $\tan\left(\frac{5\pi}{12}\right)$.

16 If $\cos 2\alpha = \sin^2\alpha$, find the exact value of $\cot\alpha$.

17 A chord of a circle has length 6 cm. If the radius of the circle is 5 cm, find the area of the minor segment cut off by the chord.

18



For the illustrated sine function, find the coordinates of the points P and Q.

19 Find the period of:

a $y = \cos\left(\dfrac{x}{3}\right)$ **b** $y = \tan(5x)$

c $y = \sin 3x + \sin x.$

20 Find the largest angle of the triangle with sides 11 cm, 9 cm, and 7 cm.

21 Find the equations of the vertical asymptotes on $[-2\pi,\ 2\pi]$ for:

a $f(x) = \csc(x)$ **b** $f:\ x \mapsto \sec(2x)$

c $g:\ x \mapsto \cot\left(\dfrac{x}{2}\right).$

22 Find the exact value of $\cos 79°\cos 71° - \sin 79°\sin 71°$.

23 Given that $\tan 2A = \sin A$ where $\sin A \neq 0$, find $\cos A$ in simplest radical form.

24 Suppose $\sin x - 2\cos x = A\sin(x + \alpha)$ where $A > 0$ and $0 < \alpha < 2\pi$. Find A and α.

25 $2\sin^2 x - \cos x = 1$ for $x \in [0,\ 2\pi]$. Find the exact value(s) of x.

26 Find x if $\arcsin(2x - 3) = -\frac{\pi}{6}$.

27 In triangle ABC, AB = 15 cm, AC = 12 cm and angle ABC measures 30°. Find the size of the angle ACB.

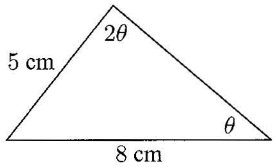
28 On the same set of axes, sketch the graphs of $f(x) = \sin x$ and $g(x) = -1 + 2f(2x + \frac{\pi}{2})$ for $-\pi \leqslant x \leqslant \pi$.

29 Find the exact value of $\arcsin(-\frac{1}{2}) + \arctan(1) + \arccos(-\frac{1}{2})$.

30 θ is obtuse and $\sin\theta = \frac{2}{3}$. Find the exact value of $\sin 2\theta$.

31 Solve for x: $\sin x + \cos x = 1$ where $0 \leqslant x \leqslant \pi$.

32



a Find $\cos\theta$. **b** Find the area of the triangle.

33 Find the exact period of $g(x) = \tan 2x + \tan 3x$.

34 Solve the equation $\cot\theta + \tan\theta = 2$ for $\theta \in \left]-\frac{\pi}{2},\ \frac{\pi}{2}\right[$.

35 If $2\theta \in \left[\pi,\ \frac{3\pi}{2}\right]$ and $\tan(2\theta) = 2$, find the exact value of $\tan\theta$.

36 In triangle PQR, PR = 12 cm, RQ = 11 cm, and $\widehat{RPQ} = 60°$. Find the length of [PQ], giving your answer in radical form.

37 Show that $\dfrac{1}{\tan\theta - \sec\theta} = -(\sec\theta + \tan\theta)$ provided $\cos\theta \neq 0$.

38 Solve for x where $x \in [-\pi,\ 3\pi]$, giving exact answers:

a $\sqrt{3}\tan\left(\frac{x}{2}\right) = -1$ **b** $\sqrt{3} + 2\sin(2x) = 0.$

39 If $\sin x = 2\sin\left(x - \frac{\pi}{6}\right)$, find the exact value of $\tan x$.

40 In a busy harbour, the time difference between successive high tides is about 12.3 hours. The water level varies by 2.4 metres between high and low tide. Tomorrow, the first high tide will be at 1 am, and the water level will be 4.7 metres at this time.

a Find a sine model for the height of the tide H in terms of time t tomorrow.

b Sketch a graph of the water level in the harbour tomorrow.

41 Find:

a $\sin\left(\arccos(-\frac{\sqrt{3}}{2})\right)$ **b** $\tan\left(\arcsin(\frac{1}{\sqrt{2}})\right)$

42 Find the exact solutions of $\sin x + \sqrt{3}\cos x = 0,\ x \in [0, 2\pi]$.

43 If $\dfrac{\sin\theta + 2\cos\theta}{\sin\theta - \cos\theta} = 2$, find the exact value of $\tan 2\theta$.

44 Without using technology, sketch the graph of $y = 2\sin\left(x - \frac{\pi}{3}\right) + 1$ for $x \in [-\pi, \pi]$.

45 Solve for x: $\cos 2x + \sqrt{3}\sin 2x = 1$ on the interval $[-\pi, \pi]$.

TOPIC 4: VECTORS

A **vector** is a quantity with both **magnitude** and **direction**.

Two vectors are **equal** if and only if they have the same magnitude *and* direction.

In examinations:

- scalars are written in italics a
- vectors are written in bold **a**.

On paper, you should write vector **a** as $\vec{a}$.

The basic unit vectors with magnitude 1 are:

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The **zero vector 0** is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

The general 3-dimensional vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

You should understand the following for vectors in both algebraic and geometric forms:

- vector addition
- vector subtraction $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- multiplication by a scalar k to produce vector $k\mathbf{v}$ which is parallel to $\mathbf{v}$
- the magnitude of vector $\mathbf{v}$, $|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$
- the distance between two points in space is the magnitude of the vector which joins them.

The **position vector** of $A(x, y, z)$ is $\overrightarrow{OA}$ or $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

Given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the distance
  $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- the midpoint of $\overrightarrow{AB}$ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

A, B, and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

The unit vector in the direction of **a** is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.

THE SCALAR OR DOT PRODUCT OF TWO VECTORS

$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between the vectors.

$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$

Properties of the scalar product:

$\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$

$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$

$(k\mathbf{v}) \bullet \mathbf{w} = k(\mathbf{v} \bullet \mathbf{w})$

$\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$

For non-zero vectors **v** and **w**:

- $\mathbf{v} \perp \mathbf{w} \Leftrightarrow \mathbf{v} \bullet \mathbf{w} = 0$
- $\mathbf{v} \parallel \mathbf{w} \Leftrightarrow |\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|$ and in this case $\mathbf{v} = k\mathbf{w}$.

The angle between vectors **v** and **w** emanating from the same point, is given by $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}$.

If $\mathbf{v} \bullet \mathbf{w} > 0$ then θ is acute.

If $\mathbf{v} \bullet \mathbf{w} < 0$ then θ is obtuse.

THE VECTOR OR CROSS PRODUCT OF TWO VECTORS

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$\mathbf{v} \times \mathbf{w}$ is perpendicular to both **v** and **w**. Its direction is found using the right hand rule.

Properties of the vector product:

If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ then $\mathbf{v} \parallel \mathbf{w}$.

$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

$(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$

Geometric properties:

$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin\theta$ where θ is the angle between the vectors.

$|\mathbf{v} \times \mathbf{w}|$ = area of parallelogram formed by vectors **v** and **w**.

$\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$ = area of triangle formed by vectors **v** and **w**.

LINES

The **vector equation of a line** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where **a** is the position vector of any point on the line, **b** is a vector parallel to the line, and $\lambda \in \mathbb{R}$.

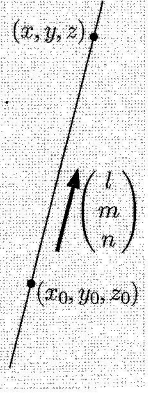
For example, if an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ for $t \geqslant 0$.

The **parametric form** for the equation of a line is $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

The **Cartesian form** for the equation of a line is $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n} = \lambda$.

The **acute angle θ between two lines** is given by $\cos\theta = \dfrac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the lines.



Lines are:

- **perpendicular** if $\mathbf{b}_1 \bullet \mathbf{b}_2 = 0$
- **parallel** if their direction vectors are parallel: $\mathbf{b}_1 = k\mathbf{b}_2$
- **coincident** if they are parallel and have a common point
- **intersecting** if you can solve them simultaneously to find a unique common point that fits both equations
- **skew** if they are not parallel and do not have a point of intersection; there are no solutions when the equations are solved simultaneously, and the lines are said to be **non-coplanar**.

The shortest distance from point P to a line with direction vector **b** occurs at the point R on the line such that $\overrightarrow{PR}$ is perpendicular to **b**.

PLANES

The **vector equation of a plane** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
where $\mathbf{r}$ is the position vector of any point on the plane,
a is the position vector of a known point on the plane,
b and **c** are non-parallel vectors that are parallel to the plane, and
$\lambda, \mu \in \mathbb{R}$ are two independent parameters.

We say this equation is in **parametric form**.

The vector $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ is a **normal vector** to the plane.

If a plane has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and passes through (x_1, y_1, z_1) then it has equation

$ax + by + cz = ax_1 + by_1 + cz_1 = d$ where d is a constant. This is the **Cartesian equation** of the plane.

If a line has direction vector **d** and a plane has normal vector **n** then the acute angle ϕ between the line and the plane is given by $\phi = \arcsin\left(\dfrac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}|\,|\mathbf{d}|}\right)$.

If two planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ then the acute angle θ between them is given by $\theta = \arccos\left(\dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$.

INTERSECTION OF PLANES

To find the intersection of two or three planes, write the system in **augmented matrix form** and use **row reduction** to reduce it to **echelon form**.

If two or more planes are **parallel** but not coincident, then there are no common points on the three planes and the system has **no solutions**.

If two planes are **coincident** and the third is not parallel, the planes meet in a line. There are an **infinite number of solutions** to the system.

If there is **no coincidence** or **parallelism**, there are three other possible scenarios:

- the line of intersection of any pair of planes is parallel to the third plane, so there are no solutions
- the three planes meet in a line, so there are infinitely many solutions
- the three planes meet in a common point, so there is a unique solution.

You should be familiar with all possible cases for the intersection of planes.

SKILL BUILDER QUESTIONS

1 Consider the *position vector* $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

- **a** Write **a** in terms of the base vectors **i**, **j**, and **k**.
- **b** Find the magnitude of **a**.
- **c** Write down a unit vector in the opposite direction to **a**.

2 On grid paper, illustrate how to find the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$ where $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Check your answer algebraically.

3
- **a** Draw a clear diagram of a parallelogram ABCD formed by two vectors $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$ along its sides.
- **b** Write the vectors $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AC}$, and $\overrightarrow{BD}$ in terms of **a** and **b**.
- **c** Given that the parallelogram is a rhombus, so $|\mathbf{a}| = |\mathbf{b}|$, calculate the product $\overrightarrow{AC} \bullet \overrightarrow{BD}$ in terms of **a** and **b**.
- **d** Using your answer to **c**, explain clearly why $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$.

4
- **a** Given $\mathbf{a} \bullet \mathbf{b} < 0$, what conclusion can you draw about the angle between **a** and **b**?
- **b** Find $\mathbf{a} \bullet \mathbf{b}$ for the vectors $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. Hence find the angle between these vectors, in degrees correct to one decimal place.

5 Find the value(s) of k for which $\begin{pmatrix} k \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ k \\ 3k \end{pmatrix}$ are:

- **a** parallel to each other
- **b** perpendicular to each other.

6
- **a** Consider the line with equation $\dfrac{x-1}{2} = \dfrac{3-y}{3} = z$.
  - **i** Find a vector parallel to the line.
  - **ii** Find a point on the line.
  - **iii** Determine whether the point $(7, -3, 2)$ lies on the line, giving reasons.
- **b** Find an equation of a line perpendicular to the line in **a**, which passes through the point $(5, -3, 2)$. Give your answer in parametric form.
- **c** Determine whether the lines in **a** and **b** intersect. If they do intersect, find the point of intersection. If they do not, state the relationship between the lines.
- **d** Find the acute angle between the line in **a** and the line L with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

7 Briefly explain how you would show that two lines in 3-D are coincident.

8 Find an equation of the plane passing through the point $(3, -1, 2)$, which is parallel to the vectors $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. Give your answer in parametric and Cartesian form.

9 **a** Find an equation of the plane passing through the points A$(-1, 2, 1)$, B$(2, 1, 3)$, and C$(4, -3, 5)$.

b Find the acute angle between the plane ABC and the line with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, $\lambda \in \mathbb{R}$.

10 Consider the plane with equation $3x - 2y + 7z = 6$. Find:

a a vector normal to the plane **b** a point on the plane

c the shortest distance from the plane to the point $(2, -1, 1)$.

11 Consider two planes with equations $2x + 4y + z = 1$ and $3x + 5y = 1$. Find:

a the acute angle between the two planes

b any solutions to the system of equations, interpreting your answer geometrically

c all points that lie on the two planes and also on the plane with equation $5x + 13y + 7z = 4$.

12 Find the distance from the point $(2, -1, 3)$ to the line

$$\frac{x-1}{2} = \frac{3+y}{3} = z.$$

13 Consider the points A$(1, -1, 2)$ and B$(5, -1, -1)$.

a Find the equation of the line L through A and B.

b Find the equation of the plane perpendicular to L, and which passes through A.

c Find a point on L which is 20 units from A.

14 Two ships A and B have paths defined by the equations

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

respectively, where distances are in kilometres and t is the time in hours.

a Find the initial position of each ship.

b Find the speed of each ship.

c Show that the two ships will pass through the same location, but not at the same time.

15 ABCD is a parallelogram with A$(-1, 2, 3)$, B$(0, 2, 4)$, and C$(1, 5, -1)$.

a Find the coordinates of D. **b** Find the area of ABCD.

16 **a** The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. If P lies on [AB] such that $\overrightarrow{\mathrm{AP}} = t\overrightarrow{\mathrm{AB}}$, show that the position vector of P is given by $\mathbf{p} = (1-t)\mathbf{a} + t\mathbf{b}$.

b For A$(2, -1, 4)$ and B$(-3, 1, -1)$, find P on [AB] such that $\overrightarrow{\mathrm{AP}} : \overrightarrow{\mathrm{PB}} = 2 : 5$.

17 Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $|\mathbf{a} - \mathbf{b}| = |\mathbf{a} + \mathbf{b}|$. Deduce that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

18 Suppose A is $(-1, 2, 1)$ and B is $(0, 1, 3)$.

a Find the equation of the line (AB) in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R}.$$

b Find the angle between (AB) and the line L defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

19 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$.

If $\mathbf{b}$ is perpendicular to $\mathbf{c} - 2\mathbf{a}$, find the value of k.

20 Find the coordinates of the point where the line

$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ meets the plane

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}.$$

21 Planes P_1 and P_2 have equations $\mathbf{r} \bullet (2\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2$ and $\mathbf{r} \bullet (\mathbf{i} - \mathbf{k}) = 5$ respectively.

a Find, to the nearest degree, the size of the acute angle between P_1 and P_2.

b Find, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, the equation of the line in which P_1 and P_2 intersect.

22 Find the coordinates of the point P where the line $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ meets the plane $3x + 2y - z = 1$.

23 Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$.

Find $(\mathbf{b} \times \mathbf{c}) \bullet 2\mathbf{a}$.

24 Consider the lines:

L_1: $x = 4 + t$, $y = 3 + 2t$, $z = -1 - 2t$
L_2: $x = -1 + 3s$, $y = 1 - 2s$, $z = 2 + s$.

a Classify the pair of lines as parallel, intersecting, coincident, or skew.

b Find the acute angle between the lines.

25 Given two vectors $\mathbf{x}$ and $\mathbf{y}$, with $|\mathbf{x}| = 2$, find the value of $|\mathbf{x} + 2\mathbf{y}|$ in the following cases:

a $\mathbf{y} = -2\mathbf{x}$

b $\mathbf{x}$ and $\mathbf{y}$ are perpendicular and $|\mathbf{y}| = 3|\mathbf{x}|$.

26 The line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is reflected in the plane

$$\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Calculate the angle between the line and its reflection. Give your answer in radians.

27 Given A$(3, -1, 5)$, B$(2, 0, -3)$, and C$(1, 3, -3)$, find $\cos \mathrm{B\widehat{A}C}$ and hence $\mathrm{B\widehat{A}C}$.

28 Find the acute angle between two diagonals of the cuboid (rectangular prism) formed by the vectors $2\mathbf{i}$, $3\mathbf{j}$, and $5\mathbf{k}$.

TOPIC 5: STATISTICS AND PROBABILITY

POPULATIONS AND SAMPLES

A **population** is an entire collection of individuals about which we want to draw conclusions.

A **sample** is any subset of the population.

A **random sample** is a sample for which each member of the population has an equal chance of being selected. Random samples are used to avoid **bias** and ensure the sample reflects the true population.

TYPES OF DATA AND ITS REPRESENTATION

Discrete data can take any of a set of distinct values $x_1, x_2, x_3, \ldots$.. It is normally **counted**.

Continuous data can take any value on a particular domain of the number line. It is normally **measured.**

Grouped data refers to data which is collected in groups or classes, often for presentation purposes. The **interval width** of a class is its upper score minus its lower score.

For both discrete and continuous data we can plot the data value on the horizontal axis and the frequency on the vertical axis.

- For discrete data we draw a **column graph**. The columns have spaces between them.
- For continuous data we draw a **frequency histogram**. There are no spaces between the columns, and the classes are all of equal width.
- A **relative frequency histogram** uses relative frequency on the vertical axis. It enables us to compare distributions of different sample size.

Data distributions may be symmetric, positively skewed, or negatively skewed.

DESCRIPTIVE STATISTICS

The **mode** is the most frequently occurring score. If there are two modes we say the data is **bimodal**. For continuous data we refer to a **modal class.**

The **median** is the middle score of an ordered sample.

The **mean** of a set of scores is their arithmetic average.

For a large population, the **population mean** μ is generally unknown. The **sample mean** $\overline{x}$ is used as an **unbiased estimate** of μ.

For ungrouped data, $\overline{x} = \dfrac{\sum\limits_{i=1}^{n} x_i}{n}$

For grouped data, we use the **mid-interval value** within each group to represent all scores within that group.

$\overline{x} = \dfrac{\sum\limits_{i=1}^{k} f_i x_i}{n}$ where f_i is the frequency of the group with mid-interval value x_i, and there are k groups.

The **variance** s_n^2 of a sample measures the **spread** of the scores about the sample mean.

The **standard deviation** s_n is the square root of the variance. You will find formulae for the standard deviation in your information booklet.

In this course, when a sample of size n is used to draw inference about a population, we use:

- the sample mean $\overline{x}$ as an unbiased estimate of μ
- the sample variance s_n^2 as an unbiased estimate of the population variance σ^2.

PROBABILITY

A **trial** occurs each time we perform an experiment.

The possible results from each trial of an experiment are called its **outcomes**. If the outcomes have the same probability of occurring, we say they are **equally likely**.

The **sample space** U is the set of all possible outcomes from one trial of the experiment.

An **event** is an outcome or set of outcomes with a particular characteristic.

EXPERIMENTAL PROBABILITY

In many situations, we can only measure the probabilities of different outcomes by experimentation.
Experimental probability = relative frequency.

THEORETICAL PROBABILITY

For an event A containing equally likely outcomes, the probability of A occurring is $\mathrm{P}(A) = \dfrac{n(A)}{n(U)}$.

For any event A, $0 \leqslant \mathrm{P}(A) \leqslant 1$.

For any event A, A' is the event that A does not occur. A and A' are **complementary events**, and $\mathrm{P}(A)+\mathrm{P}(A') = 1$.

The **union** A *or* B is $A \cup B$.

The **intersection** A *and* B is $A \cap B$.

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

For **disjoint** or **mutually exclusive** events, $\mathrm{P}(A \cap B) = 0$.

$A \cup A' = U$ and $A \cap A' = \varnothing$.

Two events are **independent** if the occurrence of each of them does not affect the probability that the other occurs. An example of this is sampling **with replacement**.

Two events are **dependent** if the occurrence of one of them *does* affect the probability that the other occurs. An example of this is sampling **without replacement**.

You should be able to use **Venn diagrams**, **tree diagrams**, **counting principles**, and **tables of outcomes** to calculate probabilities. You should also be able to use Venn diagrams to verify set identities.

CONDITIONAL PROBABILITY

For any two events A and B, $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.

For independent events, $\mathrm{P}(A) = \mathrm{P}(A \mid B) = \mathrm{P}(A \mid B')$ and $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.

BAYES' THEOREM

$$\begin{aligned} \mathrm{P}(A \mid B) &= \frac{\mathrm{P}(B \mid A)\mathrm{P}(A)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}(B \mid A)\mathrm{P}(A)}{\mathrm{P}(B \mid A)\mathrm{P}(A) + \mathrm{P}(B \mid A')\mathrm{P}(A')} \end{aligned}$$

DISCRETE RANDOM VARIABLES

A **random variable** represents the possible numerical outcomes of an experiment.

A **discrete random variable** can take any of a set of distinct values, each of which corresponds to exactly one outcome in the sample space.

For a discrete random variable where p_i is the probability of the ith outcome: $0 \leqslant p_i \leqslant 1$ and $\sum p_i = 1$.

The probabilities p_i may be given as a **probability mass function** $P(x) = \mathrm{P}(X = x)$ where x can take discrete values.

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we **expect** the event to occur is np.

The **expectation** of a random variable is $\mathrm{E}(X) = \mu = \sum\limits_{i=1}^{n} x_i p_i$.

A game with outcome X is said to be **fair** if $\mathrm{E}(X) = 0$.

The **mode** is the score with the highest frequency.

The **median** is the middle score of an ordered set.

The population **variance** is

$$\begin{aligned}\text{Var}(X) &= \sigma^2\\ &= \text{E}(X-\mu)^2\\ &= \sum (x_i-\mu)^2 p_i\\ &= \sum x_i^{\,2} p_i - \mu^2\\ &= \text{E}(X^2)-\mu^2\\ &= \text{E}(X^2)-\{\text{E}(X)\}^2\end{aligned}$$

The population **standard deviation** is $\sigma = \sqrt{\text{Var}(X)}$.

$\text{E}(aX+b) = a\text{E}(X)+b$ and $\text{Var}(aX+b) = a^2\text{Var}(X)$.

THE BINOMIAL DISTRIBUTION

The random variable X has a **binomial distribution** if there are n independent trials of the same experiment with the probability of success being a constant p for each trial. We write $X \sim \text{B}(n, p)$.

The **binomial probability distribution function** is $\text{P}(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ where $r = 0, 1, 2, ..., n$ and X is the number of successes in n trials of the experiment.

You should be able to use your calculator to find $\text{P}(X=r)$, $\text{P}(X \leqslant r)$, and $\text{P}(X \geqslant r)$.

$\text{E}(X) = \mu = np$ and $\text{Var}(X) = \sigma^2 = np(1-p)$

THE POISSON DISTRIBUTION

The random variable X has a **Poisson distribution** if the events occur singly and at random in a given interval of time or space, and the mean number m of occurrences in the given interval is finite. We write $X \sim \text{Po}(m)$.

$P(x) = \text{P}(X=x) = \dfrac{m^x e^{-m}}{x!}$ for $x = 0, 1, 2, ...$
and m is the parameter of the distribution.

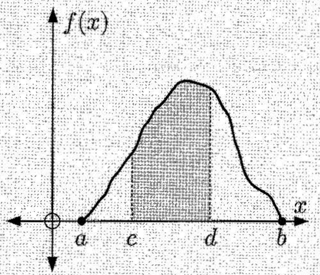
$\text{E}(X) = \mu = m$ and $\text{Var}(X) = \sigma^2 = m$.

CONTINUOUS RANDOM VARIABLES

A **continuous random variable** may have any possible *measured* value. We consider measurements that are in a particular *range* rather than an exact value. For example, we do not consider $X = 3.7$ kg but rather $X \in [3.65, 3.75[$ kg.

A **continuous probability density function** is a function $f(x)$ defined on an interval $[a, b]$ such that $f(x) \geqslant 0$ on its domain, and $\int_a^b f(x)dx = 1$.

$\text{P}(c \leqslant X \leqslant d) = \int_c^d f(x)dx$



The **mean** or **expectation** is $\text{E}(X) = \mu = \int_a^b xf(x)dx$.

The **mode** is the value of $x \in [a, b]$ for which $f(x)$ is maximised.

The **median** m is the solution of the equation $\int_a^m f(x)dx = \frac{1}{2}$.

The population **variance** is

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2)-\{\text{E}(X)\}^2\\ &= \int_a^b x^2 f(x)dx - \mu^2\end{aligned}$$

THE NORMAL DISTRIBUTION

If the random variable X has a normal distribution with mean μ and variance σ^2, we write $X \sim \text{N}(\mu, \sigma^2)$.

The probability density function is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $x \in \mathbb{R}$.

$f(x)$ is a bell-shaped curve which is symmetric about $x = \mu$.

It has the property that:

- $\approx 68\%$ of all scores lie within one σ of μ
- $\approx 95\%$ of all scores lie within two σs of μ
- $\approx 99.7\%$ of all scores lie within three σs of μ.

THE STANDARD NORMAL DISTRIBUTION

Every normal X-distribution can be transformed into the **standard normal distribution** or ***Z*-distribution** using the transformation $Z = \dfrac{X-\mu}{\sigma}$.

The standard normal distribution has mean 0 and variance 1, so $Z \sim \text{N}(0, 1)$.

The probability density function is $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ for $z \in \mathbb{R}$.

We use Z-distributions when:

- we are looking for an unknown mean μ or variance σ^2
- we are comparing scores from two different normal distributions.

You should be able to use your calculator to find normal probabilities for the situations:

- $\text{P}(Z \leqslant a)$
- $\text{P}(Z \geqslant a)$
- $\text{P}(a \leqslant Z \leqslant b)$

You should also be able to use your calculator to find the scores corresponding to particular probabilities. These scores are known as **quantiles** or ***k*-values**, and are solutions to $\text{P}(X \leqslant k) = p$ where p is known.

SKILL BUILDER QUESTIONS

1 **a** In a horse race with 10 horses in the field, why is the probability of a given horse winning *not* equal to $\frac{1}{10}$?

b Hence, explain when the rule $\text{P}(A) = \dfrac{n(A)}{n(U)}$ does not necessarily apply.

2 A bucket contains 7 red, 3 blue, 2 black, and 1 yellow ball of the same size. A random selection of 5 balls is taken from the bucket, without replacement.

a Find the probability that the random selection from the bucket will contain 2 red, 1 blue, and 2 black balls.

b Find the probability that the random selection will contain at least 1 red ball.

3 The data set 8, 5, 5, 6, 10, 7, x, 7, 8 has mean 7.

a Find the value of x.

b Find the median of the data set.

c State the mode of the data set.

d Comment on the distribution of the data.

4 $f(x)$ is the probability density function of a continuous random variable X.

a What two properties must $f(x)$ have?

b How is $\text{P}(x_1 \leqslant X \leqslant x_2)$ found?

5 **a** Tickets in a raffle are numbered 1 to 100. A ticket is drawn at random. Suppose A is the event that a ticket with number *less than* 45 is drawn, and B is the event that a ticket with number *between* 40 and 55 is drawn.

  i Are A and B mutually exclusive events? Explain your answer.

  ii Find $P(A \cup B)$.

b Suppose $P(A)$ and $P(B)$ are both non-zero. Explain why events A and B cannot be both independent and mutually exclusive at the same time.

6 A random sample of chicks were weighed one day after hatching. The results are given in the table alongside.

Weight (w g)	*Frequency (f)*
$32 \leqslant w < 36$	3
$36 \leqslant w < 40$	5
$40 \leqslant w < 44$	14
$44 \leqslant w < 48$	19
$48 \leqslant w < 52$	8
$52 \leqslant w < 56$	1

a How many chicks were weighed?

b Construct a frequency histogram to display the data.

c Use mid-interval values to estimate:

  i the mean of the data

  ii the standard deviation of the data.

7 A continuous random variable X has a probability density function defined by $f(x) = \begin{cases} a(x^2+2), & \text{for } 0 \leqslant x \leqslant 2 \\ 0, & \text{elsewhere.} \end{cases}$

a Find the constant a.

b Determine:

  i $P(0.5 \leqslant X \leqslant 1.4)$ **ii** $P(X \geqslant 1)$.

c Find the:

  i median **ii** mean **iii** variance of X.

8 It is said that events A and B are *independent* if $P(A \mid B) = P(A)$.
Use this result and the rule for conditional probability to show that this implies $P(A \cap B) = P(A)\,P(B)$.

9 A random variable X has the following distribution table:

x	-2	0	3	5
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{6}$	k	$\frac{1}{12}$

a Is the random variable discrete or continuous?

b Find k.

c Find $E(X)$, $\text{Var}(X)$, and the standard deviation of X.

d Find the median and modal values of X.

10 Events A and B are independent.
Given that $P(A \cup B) = 0.63$ and $P(B) = 0.36$, find $P(A)$.

11 38% of the students in a Year 12 IB Mathematics class are female. Of the female students in this class, 13% are left-handed, whereas 24% of the male students are left-handed.

a Find the probability that a randomly chosen student from this class is left-handed.

b Find the probability that a randomly chosen student is female, given that the student is left-handed.

12 In a sample of data, we know that $\sum f = 30$, $\overline{x} = 80.9$ and the standard deviation of this sample is 296.
Find $\sum f\,x$ and $\sum f\,x^2$ for this sample of data, giving your answers to the nearest integer.

13 How many different arrangements of the letters of the word DIPLOMA are possible if:

a there are no restrictions

b the arrangements begin and end with a vowel

c the vowels appear together?

14 A random variable X has a probability density function given by $f(x) = \begin{cases} \sin(0.5x), & \text{for } 0 \leqslant x \leqslant a \\ 0, & \text{elsewhere.} \end{cases}$

a Is the random variable discrete or continuous?

b Find the exact value of a.

c Find $E(X)$, $\text{Var}(X)$, and the standard deviation of X.

d Find the median and modal values of X.

15 Two unbiased dice are rolled and the difference between the scores is noted. Using a table of outcomes, find the probability that the difference between the scores is 3.

16 Given that $P(A) = 0.46$, $P(B) = \frac{5}{7}$ and $P(A \cup B)' = \frac{1}{12}$, find $P(A \cap B)$.

17 A typist makes on average 1 error per page. Suppose X is the number of errors made by the typist in typing a 12 page document.

a Is X a binomial or Poisson random variable?

b Find the mean and standard deviation of X.

c Find $P(X = 10)$.

d Find the probability the typist makes at least 10 errors in this document.

18 The random variable X is distributed normally with mean 37 and variance 9.

a Is the random variable discrete or continuous?

b Why is it almost impossible to have a measured score of 27?

c How many standard deviations away from the mean is the score 33?

d Explain why $P(X \geqslant 39)$ is smaller than 0.5.

e Find $P(X \geqslant 39)$.

f Find $P(31 \leqslant X \leqslant 39)$.

g Find $P(|X - 37| \leqslant 2)$.

h Find k such that $P(X \geqslant k) = 0.56$.

19 A multiple choice test consists of 30 questions with 5 answers to choose from. For each question, only one answer is correct. Let Y be the number of correct answers chosen if each answer is randomly guessed.

a Is Y a binomial or a Poisson random variable?

b Find the mean and standard deviation of Y.

c Find $P(Y = 20)$.

d Find the probability of getting a score of at least the mean plus twice the standard deviation of Y.

20 Vehicle licence plates are composed of three letters from a 26-letter alphabet, followed by a three-digit number whose first digit cannot be zero.

a How many different licence plates are possible?

b Find the probability of a randomly chosen number plate beginning with the letters AB and ending with the digit 0.

21 In the town of Expiet, 71% of the population are right-handed, 44% are either right-handed or have blonde hair but not both, and 21% do not have blonde hair.

A member of this population is selected at random. Find the likelihood that the person:

- **a** is right-handed but not blonde
- **b** is both right-handed and has blonde hair
- **c** is right-handed or has blonde hair.

22 Given $X \sim \text{N}(13, \sigma^2)$ and $\text{P}(X \leqslant 15) = 0.613$, find σ.

23 A Poisson variable X has standard deviation 3.1.

- **a** Find the mean of X.
- **b** What is the probability distribution function of X?
- **c** Find:
  - **i** $\text{P}(X = 8)$ **ii** $\text{P}(X \geqslant 11)$ **iii** $\text{P}(X \geqslant 13 \mid X \geqslant 9)$

24 Suppose X is normally distributed with $\text{P}(X \leqslant 24) = 0.035$ and $\text{P}(X \geqslant 33) = 0.262$.

Find the mean and standard deviation of X, correct to 3 significant figures.

25 A committee of 6 is chosen at random from 9 men and 8 women. Find the probability that the committee contains:

- **a** three men and three women
- **b** at least two members of each sex
- **c** an even number of women.

26 3% of a catch of prawns are damaged. If a random sample of 200 prawns is taken, estimate the mean and standard deviation of the number of damaged prawns in the sample.

27 A random variable X is Poisson with mean m, and satisfies $\text{P}(X = 1) + \text{P}(X = 2) = \text{P}(X = 3)$. Find:

a m **b** $\text{P}(X = 4)$ **c** $\text{P}(X \geqslant 2)$

TOPIC 6: CALCULUS

LIMITS

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to a, we say that $f(x)$ has a **limit** of A as x approaches a, and we write $\lim\limits_{x \to a} f(x) = A$.

In this case, $f(x)$ is said to **converge** to A as x approaches a.

A function f is said to be **continuous** at $x = a$ if and only if three conditions are all satisfied:

1. $f(a)$ is defined
2. $\lim\limits_{x \to a} f(x)$ exists
3. $f(a) = \lim\limits_{x \to a} f(a)$

We can use the idea of limits as $x \to \pm\infty$ and as $f(x) \to \pm\infty$ to find asymptotes.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

DIFFERENTIATION

The **derivative function** is $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

The **instantaneous rate of change** between two variables is given by the **gradient of the tangent** to their graph at that point.

$f'(x)$ provides:

- the rate of change of f with respect to x
- the gradient of the tangent to $y = f(x)$ for any value of x.

$f(x)$	$f'(x)$	*Name of rule*
c	0	
x^n	nx^{n-1}	
$cu(x)$	$cu'(x)$	
$u(x) + v(x)$	$u'(x) + v'(x)$	addition rule
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	product rule
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	quotient rule
$e^{f(x)}$	$e^{f(x)}f'(x)$	exponentials
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	logarithms
$\sin x$	$\cos x$	circular functions
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}, \quad x \in \left]-1, 1\right[$	inverse circular functions
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}, \quad x \in \left]-1, 1\right[$	
$\arctan x$	$\dfrac{1}{1+x^2}$	

CHAIN RULE

If $y = f(u)$ where $u = u(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

IMPLICIT DIFFERENTIATION

When differentiating it is sometimes useful to take the logarithm of both sides.

For example, if $y = \dfrac{(2x+3)^2}{\sqrt{x^2+5}}$

then $\ln y = 2\ln(2x+3) - \frac{1}{2}\ln\left(x^2+5\right)$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{4}{2x+3} - \frac{x}{x^2+5}$$

Equations expressed implicitly are not always functions. For any value of x there may be more than one value of y. Information in the question may indicate which value should be chosen.

When finding $\dfrac{dy}{dx}$ in, for example, $x^2 + xy - y^2 = 7$, a common error is to forget to differentiate the right hand side.

SECOND AND HIGHER DERIVATIVES

You should be familiar with both notations $f^{(n)}(x)$ and $\dfrac{d^n y}{dx^n}$ for higher derivatives.

TANGENTS AND NORMALS

The tangent to the curve $y = f(x)$ at the point where $x = a$, has equation

$$\frac{y - f(a)}{x - a} = f'(a).$$

The normal to the curve $y = f(x)$ at the point where $x = a$, has gradient $= -\dfrac{1}{f'(a)}$.

PROPERTIES OF CURVES

Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

$f(x)$ is **increasing** on $S \Leftrightarrow f(a) \leqslant f(b)$ for all $a, b \in S$ such that $a < b$.

$f(x)$ is **decreasing** on $S \Leftrightarrow f(a) \geqslant f(b)$ for all $a, b \in S$ such that $a < b$.

Functions which have the same behaviour for all $x \in \mathbb{R}$ are called **monotone increasing** or **monotone decreasing**.

A **stationary point** of a function is a point such that $f'(x) = 0$.

You should be able to identify and explain the significance of local and global maxima and minima, and both horizontal and non-horizontal inflections.

There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes on either side of $x = a$.

If $f''(x) \leqslant 0$ for all $x \in S$, the curve is **concave downwards** on the interval S.

If $f''(x) \geqslant 0$ for all $x \in S$, the curve is **concave upwards** on the interval S.

KINEMATIC PROBLEMS

If s represents displacement and t represents time, then

velocity $v = \dfrac{ds}{dt}$ and acceleration $a = \dfrac{d^2s}{dt^2}$.

You should understand:

- the difference between *instantaneous* velocity or acceleration, and *average* velocity or acceleration over a time period
- the difference between velocity and speed
- the physical meaning of the different combinations of signs of velocity and acceleration.

OPTIMISATION PROBLEMS

Remember that a local minimum or maximum does not always give the minimum or maximum value of the function in a particular domain. You must check for other turning points in the domain, and whether the end values of the domain give higher or lower values.

INTEGRATION

If $F(x)$ is a function where $F'(x) = f(x)$, then $F(x)$ is the **antiderivative** of $f(x)$.

FUNDAMENTAL THEOREM OF CALCULUS

For a continuous function $f(x)$ with antiderivative $F(x)$, $\int_a^b f(x)dx = F(b) - F(a)$.

AREA UNDER A CURVE

You should be able to use the idea of **upper** and **lower rectangles** to approximate the area under a curve.

You should know the difference between a definite integral and the area under a curve, but also how they are related.

If $f(x)$ is a continuous *positive* function on the interval $[a, b]$ then the area under the curve between $x = a$ and $x = b$ is $\int_a^b f(x)dx$.

The total area enclosed by $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is $\int_a^b |f(x)|\, dx$.

INDEFINITE INTEGRALS

When performing an indefinite integral, we use the rules for differentiation in reverse. Do not forget to include the **constant of integration**.

The following integrals are useful to remember:

- $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$
- $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$
- $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$
- $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$
- $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c, \; n \neq -1$
- $\int \dfrac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$
- $\int \dfrac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + c$
- $\int \dfrac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$

Trigonometric identities are often useful for integration, in particular:

- $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$
- $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$
- $\tan^2 x = \sec^2 x - 1$

INTEGRATION BY SUBSTITUTION

$\int f(u)\dfrac{du}{dx}dx = \int f(u)du$

When a function contains	*Try substituting*
$\sqrt{f(x)}$	$u = f(x)$
$\ln x$	$u = \ln x$
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$x^2 + a^2$ or $\sqrt{x^2+a^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

When using substitution to evaluate a definite integral, make sure you change the limits of integration to correspond to the new variable.

INTEGRATION BY PARTS

$\int uv'dx = uv - \int u'v\,dx$

DEFINITE INTEGRALS

- $\int_a^a f(x)dx = 0$
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

SOLIDS OF REVOLUTION

When the region enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is rotated through 2π about the x-axis to generate a solid, the volume of the solid is given by $V = \pi\int_a^b y^2 dx$.

When the region enclosed by $x = f(y)$, the y-axis, and the horizontal lines $y = a$ and $y = b$ is rotated through 2π about the y-axis to generate a solid, the volume of the solid is given by $V = \pi\int_a^b x^2 dy$.

SKILL BUILDER QUESTIONS

1 Consider the function $f(x) = \dfrac{x}{x}$.

a State the domain of $f(x)$.

b Find:

i $\lim\limits_{x\to 0^-} f(x)$ **ii** $\lim\limits_{x\to 0^+} f(x)$

c Is $f(x)$ continuous? Explain your answer.

2 **a** Find, from first principles, the derivative of $y = 3 - 2x^2$.

b Hence find the equation of the normal to $y = 3 - 2x^2$ at the point where $x = 1$.

3 Find $f'(x)$ if:

a $f(x) = \sqrt{3x^2 + 5x - 2}$ **b** $f(x) = x^2e^{2x-3}$

c $f(x) = 3^{x^2-x-2}$ **d** $f(x) = (\sin x)e^{\cos x}$

4 Find:

a $\displaystyle\int \frac{3}{x}\,dx$ **b** $\displaystyle\int \cos(3x-2)\,dx$ **c** $\displaystyle\int \frac{2x^2 - x - 3}{x^2}\,dx$

5 Find $\dfrac{dy}{dx}$ for:

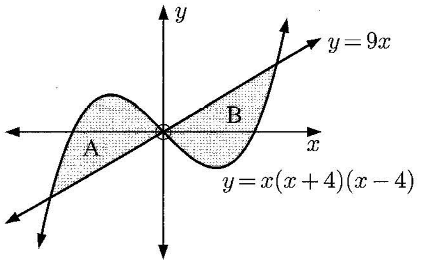
a $y = \ln(2x^2 + 8)$ **b** $y = \dfrac{x+2}{x^2+3}$

c $y = \arcsin(2x)$ **d** $y = e^{x\ln x}$

6 Find the exact value of $\displaystyle\int_3^5 \frac{x}{x^2-8}\,dx$.

7 Find the equation of the tangent to $f(x) = \dfrac{x-4}{x+2}$ at the point where $x = 3$.

8



a Find the coordinates of the points where $y = x(x+4)(x-4)$ and $y = 9x$ meet.

b Write down an integral expression for:

i area A **ii** area B **iii** area A + area B

9 Find $\displaystyle\int \sin^2 3x\,dx$.

10 Find $\dfrac{d^2y}{dx^2}$ for:

a $y = \dfrac{3}{x^2}$ **b** $y = x^2\sin(3x)$

11 Find the exact value of k, if $\displaystyle\int_0^k \frac{x}{\sqrt{x^2+4}}\,dx = 1, \ k > 0.$

12 Use integration by parts to find $\displaystyle\int x\ln x\,dx$.

13 Find the equation of the normal to $y = x^2 - 2x - 4$ at the point where $x = -1$.

14 Find the values of x for which $f(x) = \dfrac{x}{x^2-2}$ is:

a undefined **b** increasing **c** concave down.

15 Find the total distance travelled by a particle in the first 5 seconds of motion if the particle is moving in a straight line and its velocity is given by $v = t^3 - 3t^2e^{0.05t}$.

16 Find the exact coordinates and nature of the stationary points of:

a $y = xe^{-x}$ **b** $y = \dfrac{x-3}{x^2-5}$

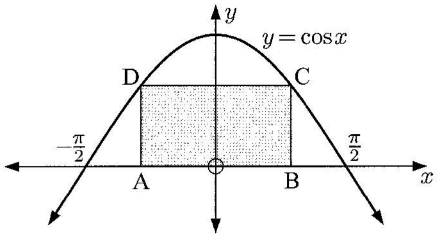
17 Find $\displaystyle\int \tan^2 2x\,dx$.

18 Find where the tangent to $y = x^3 + 2x + 1$ at the point where $x = -1$ meets the curve again.

19 **a** Find, from first principles, the derivative of $y = ax^2 + bx + c$ where a, b, and c are constants.

b *Hence* find the x-coordinate of the vertex of the general quadratic function $y = ax^2 + bx + c$.

20



Rectangle ABCD is inscribed under one arch of $y = \cos x$. Find the coordinates of C such that ABCD has maximum area.

21 Find the exact area of the region enclosed by the graph of $y = 8x - x^2$ and the x-axis.

22 A particle moves in a straight line with displacement function $s(t) = 12t - 3t^3 + 1$ cm, where $t \geqslant 0$ is in seconds.

a Find the velocity and acceleration functions for the particle's movement.

b When is the particle's:

i speed decreasing **ii** velocity decreasing?

23 Use integration by parts to find $\displaystyle\int \arctan x\,dx$.

24 Find the intervals where the function $f(x) = x^3 - 3x^2 - 9x + 5$ is:

a increasing **b** concave up.

25 Find the volume of the solid formed when the region enclosed by the graph of $y = x^2 - 2x$ and the x-axis is rotated about the x-axis. Give your answer as an exact value.

26 Find the exact value of $\displaystyle\int_0^4 \frac{1}{\sqrt{x+4}}\,dx$.

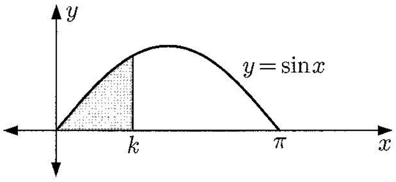
27 Find $\dfrac{dy}{dx}$ if:

a $x^2 - xy^2 + y = 21$ **b** $e^y \sin 2x = 1$

28 Integrate with respect to x:

a xe^{-x^2} **b** 5^x **c** $(x^2+1)^3$

29



The shaded region has area 0.42 units^2. Find k.

30 The kinetic energy of a moving object is given by $K = \frac{1}{2}mv^2$ where m is its mass in kg and v is its velocity in km s^{-1}.
When a rocket is fired, the three quantities K, m, and v vary. Suppose a rocket's kinetic energy is increasing at a constant rate of 50 000 units s^{-1}, while its mass is decreasing at 10 kg s^{-1} as it burns fuel.
At what rate is its velocity changing when the rocket has mass 4000 kg and velocity 8 km s^{-1}?

31 Find the exact value of the area of the region enclosed by the graphs of $y = x^2 - 3x$ and the line $y = x$.

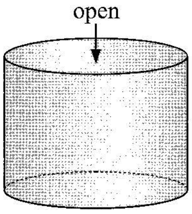
32 Find $\displaystyle\int_0^3 |x-1|\, dx.$

33 Two roadrunners start at an intersection of two roads, facing 45° apart. One runs at 28 km h^{-1} and the other at 32 km h^{-1}. Find the rate at which the distance between the runners is changing after 15 minutes.

34 **a** Show that $\int \sec x\, dx = \ln|\sec x + \tan x| + c$.

b Find $\int \sec^3 x\, dx$.

35 An open cylindrical bin is to be made from PVC plastic and is to have a capacity of 500 litres. Find the dimensions of the bin which minimise the amount of PVC plastic used.



36 Find $\int x\sqrt{3x-4}\, dx$.

37 **a** Write $\dfrac{x+2}{(x-1)(x+3)}$ in the form $\dfrac{A}{x-1} + \dfrac{B}{x+3}$.

b Hence find $\displaystyle\int \frac{x+2}{(x-1)(x+3)}\, dx.$

38 Find the volume of the solid formed when the region enclosed by the graph of $y = \ln x$, the x-axis, the y-axis, and the line $y = \ln 3$ is rotated about the y-axis. Give your answer as an exact value.

39 Use the substitution $x = 2\sin\theta$ to find $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx.$

40 Find $\int x\tan^2 x\, dx$.

41 Find the value of a such that $\displaystyle\int_0^a \frac{x^2}{x^3+1}\, dx = 2,\ a > 0.$

42 **a** Find $\displaystyle\int \frac{\sin x}{(1+\cos x)^2}\, dx.$

b Hence find the exact value of the area enclosed by the curve $y = \dfrac{\sin x}{(1+\cos x)^2}$, the x-axis, $x = \frac{\pi}{3}$, and $x = \frac{2\pi}{3}$.

43 Find $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\, dx.$

44 Find the coordinates of the point of inflection on the graph of $y = \dfrac{\ln x}{x^2}$.

45 **a** Differentiate $x^{\frac{1}{x}}$ with respect to x.

b Hence find the coordinates of the stationary point of the function $f : x \mapsto x^{\frac{1}{x}}$.

46 Find $\int x^2 \sin x\, dx$.

47 Find $\displaystyle\int \frac{1-2x}{\sqrt{1-x^2}}\, dx,\ -1 < x < 1.$

48 Find the exact values of the x-coordinates of the stationary points of the function $f : x \mapsto 2xe^x - 6e^x - 3x^2 + 12x + 5$.

49 Use integration by parts to find $\int \arccos x\, dx$.

50 Find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\, dx.$

51 A sloping wall makes an angle of 120° with the horizontal ground. A 13 metre ladder leans up against the wall. The bottom of the ladder slips, and moves at a constant speed of 2 metres per second away from the base of the wall. Find the speed at which the top of the ladder moves down the wall when the foot of the ladder is 7 metres from the base of the wall.

52 Find $\displaystyle\int \frac{\arctan x}{1+x^2}\, dx.$

53 Use the substitution $x = 2\sin\theta$ to find $\int \sqrt{4-x^2}\, dx$.

Exam Set 1 NO CALCULATORS

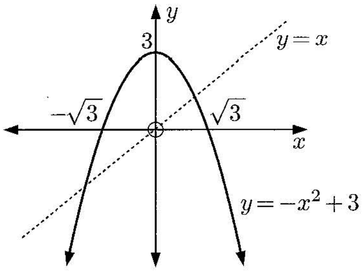
1 **a** Find $\displaystyle\sum_{r=1}^{3} (2r + 2^r)$.

b Find an expression for $\displaystyle\sum_{r=1}^{n} (2r + 2^r)$.

2 If $\log_a 2 = b$ and $\log_a 3 = c$, express $\log_a \sqrt{72}$ in terms of b and c.

3 Find the exact value of $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\cos^2 x + \tan^2 x)\, dx.$

4 **a** On the graph below, sketch the reflection of $y = f(x) = -x^2 + 3$ in the line $y = x$.



b Briefly explain why the reflection you have drawn is not the inverse function of $y = f(x)$.

c Find the equation of the inverse of $x \mapsto -x^2 + 3,\ x \leqslant 0$ and illustrate both the function and its inverse.

5 Factorise $f(x) = 2x^3 - x^2 - 8x - 5$ into real factors, and hence determine the values of x for which $f(x) > 0$.

6 The radius of a circle is 4 cm. Find the exact value of the area of the minor segment of the circle cut off by a chord of length $4\sqrt{3}$ cm.

7 In triangle PQR, $\sin \widehat{PRQ} = \frac{3\sqrt{3}}{14}$, QP = 3 cm, and PR = 7 cm. Find the possible values of $\widehat{PQR}$, giving your answer in degrees.

8 Find the exact value of $\sin\theta$ if $3\cos 2\theta + 2 = 7\sin\theta$.

9 Find the exact value of $\cos\left(\arcsin\frac{3}{5} + \arccos\frac{2}{3}\right)$.

10 Consider the points $X(3\mu, 2, 1)$ and $Y(\mu, 1-3\mu, 2\mu-1)$, where μ is a constant.

If O is the origin, find all values of μ for which $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{OY}$.

11 Find $\displaystyle\int \frac{x^2}{\sqrt{x+2}}\,dx$.

12 If the polynomial $x^n + ax^2 - 6$ leaves a remainder of -3 when divided by $(x-1)$, and a remainder of -15 when divided by $(x+3)$, find the values of a and n.

13 Consider the function $f : x \mapsto a + \dfrac{b}{x+c}$.

The graph of f has asymptotes $x = -2$ and $y = 3$, and passes through $(2, 4)$.

- **a** Find the values of a, b, and c.
- **b** State the domain and range of f.
- **c** Find f^{-1} and state the domain and range of f^{-1}.

14 Find k if $\displaystyle\int_1^k 3\sqrt{10-x}\,dx = 38$.

15 Each of two circles with the same radius passes through the centre of the other. Find the area common to both circles if the radii are 6 cm, giving your answer as an exact value.

16 The graph of $x \mapsto kx^2 - 3x + (k+2)$ cuts the x-axis in two distinct points. Find the set of possible real values of k.

17 $1 - 2i$ is a zero of the polynomial $2z^3 - 9z^2 + 20z - 25$. Find the other two zeros.

18 Find all values of k for which this system of equations has a solution other than the origin $(x, y, z) = (0, 0, 0)$.

$$\begin{cases} 2x - 2y + kz = 0 \\ x + 4z = 0 \\ kx + y + z = 0 \end{cases}$$

19
- **a** Prove that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.
- **b** Hence, or otherwise, find the exact value of the period of the function $f(x) = \sin 5x\cos 2x$.
- **c** Find $\int \sin 5x\cos 2x\,dx$.
- **d** Hence find the exact value of $\int_0^{\frac{\pi}{3}} \sin 5x\cos 2x\,dx$.

20 [AB] is a diameter of a circle of radius 5 cm, and P is a point on the circle. P moves at a constant rate in a clockwise direction on the circle, and completes one rotation every 10π seconds. Find the rate of change in the area of triangle ABP when $\widehat{PAB} = \frac{\pi}{3}$ and P is moving towards B.

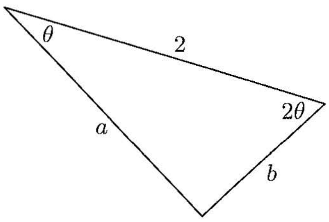
21 Use the substitution $x = \sin u$ to find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

22 ABCD is a rectangle such that A and B are points on the graph of $y = \sqrt{1-x^2}$, and C and D lie on the x-axis. Let C have coordinates $(a, 0)$.

- **a** Write an expression for the area of ABCD.
- **b** Find the exact value of a when the area is a maximum.
- **c** Find the maximum area of the rectangle.

23 Two dice are rolled. The score is the smaller of the two numbers that appear. If the same number appears on both dice, then the score is that number. What is the probability that the score is 4?

24 In the triangle shown, find a in terms of b.



25 Prove that for all $n \in \mathbb{Z}^+$, $n > 1$,

$$\int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx = \frac{1}{n-1}.$$

Exam Set 2 CALCULATORS

1 A geometric series with common ratio r has second term 6 and a sum of 49. It is known that $\frac{1}{2} < r < 1$. Find:

- **a** r **b** u_n, the nth term of the series.

2 Eight different books are to be placed in a row on a bookshelf. Three of the books are mathematics books. How many arrangements are possible if:

- **a** there are no restrictions
- **b** the mathematics books are always placed together
- **c** there has to be a mathematics book at each end of the shelf?

3 A sector cuts off an angle of 53° at the centre of a circle of radius 7 cm. Find:

- **a** the perimeter **b** the area of the sector.

4 Find the constant term in the expansion of $\left(2x^2 + \dfrac{1}{x}\right)^9$.

5 Find the points of intersection of the graphs of $x^2y = 4 + x$ and $y = e^x - 3x + 1$, where $-5 \leqslant x \leqslant 5$.

6 For what values of x in the interval $[0, 2\pi]$ is $\arccos(\sin 3x + \cos 2x)$ defined?

7 A receptionist walks to work every day.
If it is not raining, the probability that he is late is $\frac{1}{6}$.
If it is raining, the probability that he is late is $\frac{3}{5}$.
The probability that it rains on a particular day is $\frac{1}{5}$.
On one particular day the receptionist is late. Find the probability that it was raining on that day.

8 X is a binomial random variable for which the number of trials is 7, and the probability of success of each trial is p. Find the possible values of p if $P(X = 4) = 0.25$.

9 Find a unit vector parallel to $\lambda\mathbf{i} + \mathbf{j} - \lambda\mathbf{k}$ and perpendicular to $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

10 In a school, $\frac{2}{7}$ of the students travel to school by bicycle.
Seven students are chosen at random.
Find the probability that exactly 4 of them travel to school by bicycle.

11 A box contains 1000 fish with 3.7% not suitable for sale. If 20 fish are chosen at random from the box, find the probability that:

- **a** all of them are suitable for sale
- **b** exactly one of them is not suitable for sale.

12 The coefficient of x^4 in the expansion of $(ax+3)^5$ is equal to the coefficient of x^5 in the expansion of $(ax+3)^7$. Find a.

13 Consider the functions $f(x) = \dfrac{1}{x+5}$, $x \neq -5$ and $g(x) = 3x$.

a Calculate $(g \circ f)(4)$ as a rational number.

b Find $g^{-1}(x)$.

c Find the domain of g^{-1}.

14 The graph of the function $f : x \mapsto ax^3 + bx^2 + cx + d$ has a maximum turning point at $(0, 1)$ and a minimum turning point at $(-2, -2)$. Find a, b, c, and d.

15 A police radar gun measuring the speeds of cars on the road is known to produce errors at random that are distributed normally with mean 0 and standard deviation σ.
We say the error E is distributed normally as

$$E \sim \mathrm{N}(0, \sigma^2).$$

The police will not fine a driver without being 99% confident that a car is travelling at greater than 60 $\mathrm{km\,h^{-1}}$ in a 60 $\mathrm{km\,h^{-1}}$ zone.
Suppose they are 99% sure that a car is travelling at greater than 60 $\mathrm{km\,h^{-1}}$ when they record a speed of more than 65 $\mathrm{km\,h^{-1}}$ on the radar gun.

a Explain why the distribution of readings for a car travelling at 60 $\mathrm{km\,h^{-1}}$ is given by the random variable $X = 60+E$.

b Find the standard deviation σ of the distribution of errors given by the radar gun.

16 If $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, show that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$.

17 Find the exact value of p if $p > 0$ and $\displaystyle\int_0^p (x^3+x)\,dx = \tfrac{15}{4}$.

18 Let f and g be two functions such that $g(x) = 3x-2$ and $(f \circ g)(x) = x+2$. Find $f(9x-8)$.

19 The diameters of discs produced by a machine are normally distributed with mean 73 mm and standard deviation 1.1 mm. Find the probability of the machine producing a disc with a diameter larger than 75 mm.

20 Independent events A and B are such that $\mathrm{P}(A) = 0.35$ and $\mathrm{P}(A \cup B) = 0.75$. Find $\mathrm{P}(B)$.

21 A couple is told that the probability that they will have blonde haired children is $\frac{1}{7}$. The couple would like to have 5 children.

a What is the expected number of blonde children?

b Find the probability that 3 of the 5 children will have blonde hair.

c Find the probability that more than 3 of the 5 children will have blonde hair.

22 **a** Find the exact value of the area enclosed by the curve $y = \dfrac{1}{\sqrt{4-x^2}}$, the y-axis, the x-axis, and the line $x = 1$.

b The region in **a** is rotated about the x-axis. Find the volume of the solid generated.

23 When a biased die is rolled, the numbers from 1 to 6 appear according to the following probability distribution.

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{1}{7}$	y

a Find the value of y.

b Find the exact values of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

c Write a brief interpretation of the values found in part **b**.

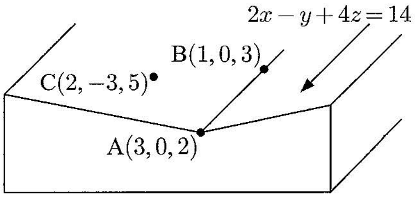
24 **a** Prove that $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$.

b Prove by mathematical induction that, for all $n \in \mathbb{Z}^+$,

$$\sin x + \sin 3x + \sin 5x + \ldots. + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$$

25 **a** Given the points A(3, 0, 2), B(1, 0, 3), and C(2, −3, 5), show that $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.

b Students are using equations of planes to model two hillsides that meet along a river. The river is modelled by the line where the two planes meet.



One of the hillsides contains the points with coordinates A(3, 0, 2), B(1, 0, 3), and C(2, −3, 5).
The other hillside has an equation $2x - y + 4z = 14$.

i Show that A and B are two points on the river.

ii Show that an equation for the river is

$$\mathbf{r} = \begin{pmatrix} 3-2\lambda \\ 0 \\ 2+\lambda \end{pmatrix} \quad \text{for } \lambda \in \mathbb{R}.$$

iii Show that B is the closest point on the river to C.

iv Find the shortest distance from the river to C.

Exam Set 3 NO CALCULATORS

1 Let $f(x) = x^2 + 4x$ for $-\infty < x \leqslant -2$ and $g(x) = \sqrt{3-2x}$.
Find: **a** $f^{-1}(x)$ **b** $(g \circ f)(-3)$.

2 In triangle ABC, $\mathrm{A\widehat{B}C} = 2\theta$, $\mathrm{A\widehat{C}B} = \theta$, AB = 3 cm, and AC = 5 cm. Find the exact value of the area of the triangle.

3 The probability that a salesperson leaves her mobile phone in a shop is $\frac{1}{6}$. After visiting two shops in succession, the salesperson discovers her mobile phone is missing. What is the probability that she left the mobile phone in the first shop?

4 Find the exact value of $\cot A$ if $\cot 2A = \frac{3}{5}$ and A is obtuse.

5 **a** For what real values of m will the line $y = mx + 16$ be a tangent to the parabola $y = x^2 + 25$?

b Given $f(x) = x^2 + x(2-k) + k^2$, find the possible real values of k for which $f(x) > 0$ for all real values of x.

6 Solve $\log_3\left(4x^2 - 5x - 6\right) = 1 + 2\log_3 x$.

7 Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$.

8 The velocity of a particle, in metres per second at time t seconds, is given by $v = t^3 - 3t^2 + 2t$. Calculate the distance travelled by the particle in the first 3 seconds of motion.

9 Find the exact values of θ if $\theta \in [0, 2\pi]$ and $\cos\theta + \sin\theta = \sqrt{2}$.

10 Find the exact value of the x-coordinate of the stationary point of the function $f : x \mapsto e^{x \ln x}$, $x > 0$.

11 An infinite geometric series with common ratio r, $0 < r < 1$ is such that the sum of the first 3 terms is half the sum of the series. Find the exact value of r.

12 $z = 2 - i$ is a solution of the equation $z^3 - 6z^2 + 13z - 10 = 0$. Find the other two solutions.

13 For these functions state the domain and range:

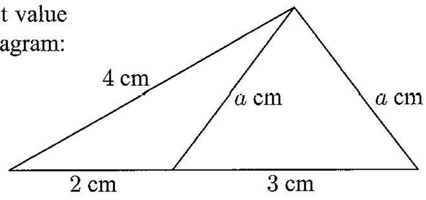
a $x \mapsto \sqrt{5-x}$ **b** $x \mapsto$ "distance from nearest integer"

14 **a** Find $\displaystyle\int \frac{1+x}{4+x^2}\,dx$.

b Use integration by parts to find $\int x^2 \ln x \, dx$.

15 **a** Simplify $\sin(\arcsin a + \arcsin b)$, giving your answer in terms of a and b.

b Find the exact value of a in the diagram:



16 Find the exact value of the x-coordinates of the stationary points of the function $f : x \mapsto \log_3\left(\dfrac{x^2+1}{3x+1}\right)$.

17 The equations of two lines are:

L_1: $x = 3\lambda - 4$, $y = \lambda + 2$, $z = 2\lambda - 1$

L_2: $x = \dfrac{y-5}{2} = \dfrac{-z-1}{2}$.

a Determine the point of intersection of L_1 and the plane $2x + y - z = 2$.

b Find the point of intersection of L_1 and L_2.

c Find an equation for the plane that contains L_1 and L_2.

18 The region enclosed by $y = \arccos x$, $y = \frac{\pi}{3}$, $y = \frac{\pi}{6}$, and the y-axis is rotated through 2π about the y-axis. Find the volume of the solid generated.

19 Sketch the graph of $f : x \mapsto 3 + \dfrac{2}{x+1}$, $-5 \leqslant x \leqslant 5$, $x \neq -1$, showing clearly all asymptotes.

On the same set of axes, sketch the graphs of $|f(x)|$ and $\dfrac{1}{f(x)}$.

20 $\cos x + \sin x \cos x + \sin^2 x \cos x + ...$ is an infinite series defined for $0 < x < \frac{\pi}{2}$.

a Find the sum of the series.

b Determine the values of x for which the sum of the series is $\sqrt{3}$.

21 **a** Prove that $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$.

b Find $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$ for $0 < x < \frac{\pi}{2}$.

c Hence prove that $\int \sec x \, dx = \ln(\sec x + \tan x) + c$ for $0 < x < \frac{\pi}{2}$.

d Find the exact value of the area enclosed by the graph $y = \sec x$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$.

22 Find the equation of the tangent to the curve $3x^2 - 2y^2 = 10$ at the point where $x = 2$ and $y < 0$.

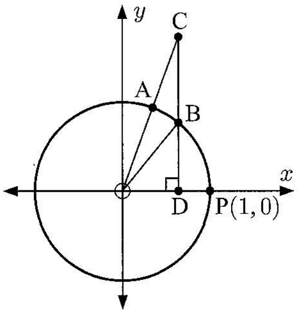
23 A random variable X has probability density function $f(x)$ where

$$f(x) = \begin{cases} \frac{1}{3}x, & 0 \leqslant x < 1 \\ \frac{1}{3}, & 1 \leqslant x < 2 \\ \frac{1}{12}(6-x), & 2 \leqslant x < a \\ 0, & \text{elsewhere.} \end{cases}$$

Find: **a** the value of a **b** the median value of X.

24 Find the coordinates of the points where the tangents to the curve $6x^2 + 4xy + 2y^2 = 3$ are horizontal.

25 On the unit circle shown, $\widehat{AOP} = \alpha$ and $\widehat{BOP} = \beta$. [OA] is produced to meet the line through B parallel to the y-axis at C. [CB] is produced to meet the x-axis at D.



a Write down the lengths of [OD] and [BD].

b Show that $\text{OC} = \dfrac{\cos\beta}{\cos\alpha}$.

c Show that $\text{BC} = \cos\beta\tan\alpha - \sin\beta$.

d Use the sine rule in triangle OBC to show that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

e Hence find the exact value of $\sin 15°$.

Exam Set 4 CALCULATORS

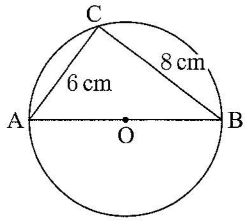
1 Consider the arithmetic series $3 + 8 + 13 + ...$

a Find an expression for S_n, the sum of the first n terms of the series.

b Find the smallest value of n for which the sum of the series is greater than 1000.

2 When $(1 + ax)^n$, $n \in \mathbb{Z}^+$ is written in expanded form as far as the third term, the result is $1 + 35x + 525x^2$. Find a and n.

3 In the diagram, [AB] is the diameter of the circle, $\text{AC} = 6$ cm, and $\text{BC} = 8$ cm. Find the area of the minor segment cut off by [AC].



4 A discrete random variable takes the values $X = 0, 2, 7$ with probabilities $\frac{1}{4}$, $\frac{3}{7}$, and k respectively.

a Find k.

b Find the mean, median, and mode of X.

c Find the variance and standard deviation of X.

5 **a** Find the equation of the normal to the graph of $y = \sin x$, at the point where $x = \frac{\pi}{6}$.

b Find the coordinates of the point where the normal in **a** meets the x-axis.

6 For what values of a does the function $f(x) = \dfrac{ax+2}{x^2+1}$ have stationary points?

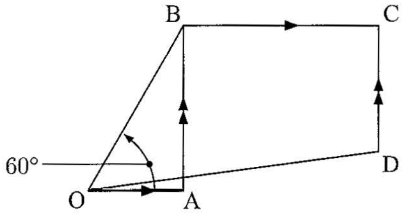
7 **a** Find the exact area of the region enclosed by the curve $y = xe^{-0.1x^2}$, the x-axis, and $x = 4$.

b Find the volume generated when the region in **a** is rotated through 2π about the x-axis.

8 A particle starts from O with an initial velocity of 5 m s^{-1}. The acceleration of the particle after t seconds is given by $a = 3t^2 - 2t + 1$.
Find an expression for the displacement function $s(t)$.

9 Find the exact value of $\sin 2(\arcsin \frac{3}{5})$.

10 In the figure below, $\overrightarrow{OB}$ is three times longer than $\overrightarrow{OA}$. $\overrightarrow{BC}$ is parallel to $\overrightarrow{OA}$ and twice its length. $\overrightarrow{CD}$ is parallel to $\overrightarrow{BA}$. Angle AOB is 60°.



Suppose A and B have position vectors **a** and **b** respectively, relative to O, and that $\overrightarrow{CD} = k(\overrightarrow{BA})$.

a Determine vector expressions for:

i $\overrightarrow{BA}$ **ii** $\overrightarrow{OD}$

b Find k if $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{AB}$.

11 If $y = \ln\left(x^2 - 3\right)$, simplify $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2$.

12 Consider the graph of the function $f : x \mapsto \dfrac{e^{x^2}}{e^x - 1}$.

a Find the equation of any vertical asymptotes, giving your answer in exact form.

b Find the coordinates of the turning points.

c Sketch the graph of $y = \dfrac{e^{x^2}}{e^x - 1}$ for $x \in [-1, 2]$.

13 Given the vectors $\mathbf{a} = -2\mathbf{i} + p\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3(p+4)\mathbf{j} + (2p-5)\mathbf{k}$, find all values of p for which **a** is perpendicular to **b**.

14 Find the term independent of x in the expansion of $\left(\dfrac{5x^2}{2} - \dfrac{2}{5x}\right)^{12}$.

15 Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \dfrac{x}{8} + c, & 0 \leqslant x \leqslant 2 \\ 0, & \text{elsewhere.} \end{cases}$$

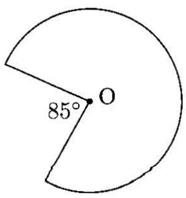
Find:

a the value of c **b** the expected value of X

c the standard deviation of X.

16 Consider two complex numbers $z = 5 - 3i$ and $w = b + 2i$, $b \in \mathbb{Q}$. Find b such that $\dfrac{z}{w} \in \mathbb{R}$.

17 The diagram alongside shows a major sector of the circle with centre O. The perimeter of the sector is 100 cm. Find the area of the sector.



18 Find $\int \sin^3 x \, dx$.

19 In a maths quiz there are 30 multiple choice questions with five alternative answers for each question. Only one of the answers is correct in each case.

a Suppose you guess every question and you answer all the questions. Find the probability of obtaining:

i exactly 20 correct answers

ii at least 15 correct answers

iii no more than 25 correct answers.

b Now suppose you have studied hard and have an 85% chance of answering each question correctly. Find the probability of obtaining:

i exactly 20 correct answers

ii at least 15 correct answers.

20 A survey of 100 people is conducted to find out how long people spend travelling to work. The following results were recorded:

Travelling time (t min)	*Frequency*
$0 \leqslant t < 10$	11
$10 \leqslant t < 20$	19
$20 \leqslant t < 30$	32
$30 \leqslant t < 40$	22
$40 \leqslant t < 50$	9
$50 \leqslant t < 60$	7

a Draw a frequency histogram to display the data.

b Find the modal class of the data.

c Use mid-interval values to estimate the mean and standard deviation for the data.

21 The table below shows the number of items produced by machines I and II, and the probability of each machine producing a faulty item.

Machine	*Number of items made*	*Probability of making faulty item*
I	1700	5.2%
II	3300	4.3%

a If an item is chosen at random from the total number of items produced, what is the probability that it is faulty?

b If an item is selected at random and it is found to be faulty, what is the chance that it is produced by machine II?

22 **a** A machine produces bags of sugar whose weights are distributed normally with mean 120 g and standard deviation 1.063 g. If the weight of a bag of sugar is less than 118 g, the bag is rejected.
Find the percentage of bags that are rejected, correct to 3 decimal places.

b The settings of the machine are now altered. It is found that 6% of the bags are rejected, but the mean has not changed. Find the new standard deviation, correct to 3 decimal places.

c The settings of the machine are adjusted once more. The new value of the standard deviation is maintained. Find the value, correct to two decimal places, at which the mean must be set, so that only 3% of the bags are rejected.

d With the new settings from part **c**, it is found that 70% of the bags of sugar have a weight which lies between x g and y g, where x and y are symmetric about the mean. Find the values of x and y, giving your answers correct to two decimal places.

23 Consider the graph of the function $y = f(x)$ where

$$f(x) = \frac{x-2}{x^2+cx-6}.$$

a Find the axes intercepts.

b Find an expression for $f'(x)$.

c For what values of c does the function have at least one stationary point?

d For $c = 0$, find the coordinates of the point of inflection of the curve, giving your answer to 3 significant figures.

e For $c = -4$, find the exact value of

$$\int_{-1}^{1} \frac{x-2}{x^2+cx-6}\,dx.$$

24 Let P$(u, \cos u)$ be any point on the graph of $y = \cos x$ in the first quadrant.

a Find the equation of the tangent to the curve $y = \cos x$ at P.

b The tangent at P meets the x-axis at A and the y-axis at B. Find the coordinates of A and B.

c Find an expression for the area of triangle AOB.

d Find the coordinates of P for which the area of triangle AOB is a minimum.

25 Consider the lines L: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

and M: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.

a Find the acute angle between the lines L and M.

b Find a vector **n** that is perpendicular to both lines.

c Find an equation of the plane P that contains the line L and which is perpendicular to **n**.

d Show that a vector $\overrightarrow{AB}$, where A $\in L$ and B $\in M$, is of the form $\begin{pmatrix} -1-\lambda+2\mu \\ 3-2\lambda-\mu \\ -2+\lambda+\mu \end{pmatrix}$.

e Find the values of λ and μ such that $\overrightarrow{AB}$ is parallel to **n**. Hence, or otherwise, find the distance between the lines L and M. Comment on the nature of these two lines.

Exam Set 5 NO CALCULATORS

1 **a** Find real numbers a and b such that $(a+2i)(b-i) = 17+7i$.

b Find p and q where $p, q \in \mathbb{R}$, $p > 0$, such that $(p+qi)^2 = -3+6i\sqrt{6}$.

2 Find the exact value of $\tan x$ if $2\sin\left(x+\frac{\pi}{6}\right) = \sin x$.

3 Find the exact value of x if $\frac{6}{7^x} - 2(7^x) = 1$.

4 **a** Find the x-coordinate of the stationary point of $y = (x-3)e^{2x}$.

b For what values of x is the graph of the function $f(x) = x^2e^x$ concave down?

5 Find the largest set of values of x such that the following functions take real values.

a $f(x) = \sqrt{x^2-1}$

b $g(x) = \ln(1-x^2)$

c $F(x) = \frac{|x-1|}{x}$

d $G(x) = \sqrt{\frac{2x-3}{x+2}}$

6 Consider the function $g : x \mapsto 2x^2 - 3x + \sin x$. Find simplified expressions for the functions f and h where the graphs of f and h are reflections of g in the x and y-axes respectively.

7 If $\sin A = \frac{3}{4}$ and $\cos B = \frac{2}{3}$ where A and B are acute, find the exact value of $\sin(A-B)$.

8 In triangle ABC, $\widehat{ABC} = 60°$, AB $= 5$ cm, BC $= 2$ cm, and AC $= k\sqrt{3}$ cm. Find the value of k.

9 Find the equation of the tangent to the curve $x^3y^2 - xy + y = 4$ at the point where $x = 1$ and $y > 0$.

10 Find $\int x\sqrt{x-3}\,dx$, $x > 3$.

11 Use integration by parts to find $\int \arcsin x\,dx$.

12 A particle moving in a straight line has initial displacement $s = 3$ metres. Its velocity in metres per second at time t seconds is given by $v = 2s$. Show that the displacement function $s = 3e^{2t}$ metres satisfies these conditions.

13 Find the exact value of $\int_0^{\frac{\pi}{4}} \left(\tan x - \frac{\tan x - 1}{\tan x + 1}\right) dx$.

14 Find the exact value of $\tan x$ if $\sin x = \cos(x + \frac{\pi}{3})$.

15 The sides [AB], [BC], and [AC] of triangle ABC have lengths 5 cm, 3 cm, and 7 cm respectively.

a Find $\cos \widehat{ABC}$.

b Hence find the area of the triangle.

16 If $\sin\gamma = \frac{\sqrt{3}}{7}$, find the value of $\cos 2\gamma$.

17 Calculate the area of the region enclosed by the curve $x = 8y - 7 - y^2$ and the line $x - 2y + 2 = 0$.

18 The displacement of a particle moving in a straight line is given by $s = 4e^{0.2t} - e^{0.3t} + 10$ m, where t is the time in seconds. Find:

a the initial displacement of the particle

b the initial velocity of the particle

c the exact time at which the particle first comes to rest.

19 **a** By expanding $(\cos\theta + i\sin\theta)^5$ and using De Moivre's theorem, find an expression for $\sin 5\theta$ in terms of $\sin\theta$.

b Hence show that the exact value of $\sin 36°$ is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$.

20 In triangle ABC, BC $= a$, AC $= b$, and AB $= c$. D is the midpoint of [BC], and the length of the median [AD] is m.

a Show that $m^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$.

b Hence show that the area of an isosceles triangle with equal sides x units and base a units is given by $\frac{1}{4}a\sqrt{4x^2 - a^2}$.

21 If $\tan\alpha = \frac{3}{\sqrt{5}}$ and α lies in the interval $]\pi, \frac{3\pi}{2}[$, find the exact value of $\sin 2\alpha$.

22 The region enclosed by the graph of $y = 1 + \tan x$, the x and y-axes, and $x = \frac{\pi}{4}$ is rotated through 2π about the x-axis. Find the volume of the solid generated.

23 Adam, Amy, Ben, and Bianca are asked to sit down in a line.

- **a** In how many different ways can the seats be chosen?
- **b** How many different seating arrangements result in each person sitting next to someone with the same first letter of their name?
- **c** If the seats are chosen at random, find the probability that Adam sits next to Ben.

24

- **a** Show that the plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 5$ contains the line

  L_1: $\mathbf{r} = (-2t + 2)\mathbf{i} + t\mathbf{j} + (3t + 1)\mathbf{k}$, $t \in \mathbb{R}$.
- **b** Find k when the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix} = 3$ contains L_1.
- **c** Without using row operations, find the values of p and q for which the following system of equations has an infinite number of solutions. Clearly explain your reasoning.

$$\begin{aligned} 2x + y + z &= 5 \\ x - y + z &= 3 \\ -2x + py + 2z &= q \end{aligned}$$

25

- **a** Use mathematical induction to prove that

  $\cos\theta + \cos 3\theta + ... + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$

  for $n \in \mathbb{Z}^+$, $\sin\theta \neq 0$.
- **b** Expand $(\cos\theta + i\sin\theta)^3$.
- **c** Using De Moivre's theorem, find an expression for $\cos 3\theta$ in terms of $\cos\theta$.
- **d** Find an expression for $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.
- **e** Find the exact value of $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin 6\theta}{2\sin\theta}\, d\theta$.

Exam Set 6 CALCULATORS

1 Find the sum of the first 100 terms of the series $\ln\sqrt{2} + \ln 2 + \ln\sqrt{8} + ...$, giving your answer in the form $n \ln 2$ where $n \in \mathbb{Z}$.

2 Find the area enclosed by the curve $y = ax^2$, $a > 0$, and the line $y = x$.

3 Use integration by parts to find $\int x^2 e^x dx$.

4 Solve for x: $\ln(x^2 + 9) - 2 = \ln|x + 5|$.

5 Let Y be a binomial random variable with mean 3 and standard deviation $\frac{3}{2}$. Find $\text{P}(Y \leqslant 4)$.

6 After t seconds, the velocity of a particle moving in a straight line is given by $v = \dfrac{2t}{4 + t^2}$ m s^{-1}.

- **a** Find an expression for the displacement s of the particle, if the initial position is $s = -3$ m.
- **b** Find an expression for the acceleration a of the particle after t seconds. Simplify your answer.

7 Consider $f(x) = e^{-\frac{1}{2}x^2}$ and $g(x) = e^{\frac{1}{2}x^2} - 1$ for $0 \leqslant x \leqslant 1.5$. Find the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = g(x)$, correct to three decimal places.

8 Suppose h and k are the values of x on the interval $-1 < x < 2.5$ for which $f(x) = \ln|4x - x^3|$ is not defined.

- **a** Find the *exact* values of h and k.
- **b** Sketch the graph of $f(x)$ on the given interval, indicating the number of zeros of $f(x)$. Show also the position of any asymptotes.
- **c** Find all the zeros of $f(x)$.

9 The hour hand and minute hand of a large railway station clock are 2 metres and 3 metres long respectively. Find the rate at which the distance between the ends of the hands is changing at 4.00 pm.

10 Find the area enclosed by the graphs of $y = \dfrac{4}{1 + x^3}$ and $y = 4(x-1)^4$ in the first quadrant.

11 A farm produces 200 tonnes of grain in its first year of operation, and the yield improves by 3% each year thereafter.

- **a** How long will it take for the annual crop to double?
- **b** Find, to the nearest tonne, the total yield of crops over the first eight years.

12 A sector of a circle has area 15 cm^2 and perimeter 16 cm. Find the radius of the circle and the angle of the sector at the centre of the circle.

13 The area of triangle ABC is 23 cm^2. If BA = 10 cm and AC = 8 cm, find two possible values for the length of [BC].

14 The roots of the equation $x^2 - kx + 4 = 0$ are α and β. Find, in terms of k:

- **a** $\alpha^2 + \beta^2$
- **b** a quadratic equation whose roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

15 A box contains two dice with faces that are numbered 1 to 6. One of the dice is fair, and the other is weighted so that the probability of a one appearing is 0.5.

- **a** If a die is selected at random and tossed, what is the probability that a one appears?
- **b** If a one appears, what is the probability that the chosen die was the weighted one?

16 Find the positive integer n such that the coefficients of x^2 in the binomial expansions of $(1 + x)^{2n}$ and $(1 + 15x^2)^n$ are equal.

17 Smith and Co. produce jars of jam with net weight that is normally distributed with mean 475 g and standard deviation 7.5 g. What percentage of jars have a net weight of less than 460 g?

18 Find the acute angle between the two planes with equations

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$$

$$\mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \ s, t \in \mathbb{R}.$$

19 The random variable Y follows a Poisson distribution such that $(\text{E}(Y))^2 = 2\,\text{Var}(Y) + 3$. Find $\text{P}(Y \geqslant 3)$.

20 Let $f(x) = \sin 2x + \sin 4x$

- **a** Sketch the graph of $y = f(x)$ for $0 \leqslant x \leqslant \pi$, clearly showing the axes intercepts.
- **b** Show that the line $y = \frac{4}{\pi}x$ cuts the curve at the origin and at $(\frac{\pi}{4}, 1)$.
- **c** Find the exact value of the area of the region enclosed by the line $y = \frac{4}{\pi}x$ and the curve $y = f(x)$ for $0 \leqslant x \leqslant \pi$.

21 In a group of 45 students at an international school, 25 hold an American passport, 15 hold an Australian passport, and 8 have neither an American nor an Australian passport. Draw a Venn diagram to illustrate this situation.

If a student is selected at random, find the probability that the student:

- **a** has both an American and an Australian passport
- **b** holds neither an American nor an Australian passport
- **c** holds exactly one of an American or an Australian passport.

22 The points A(2, −1, 3), B(6, 1, 1), and D(7, 5, 6) are three vertices of the parallelogram ABCD. Find the area of this parallelogram.

23 A zoologist knows that the lengths of a certain species of tropical fish are normally distributed with mean length m cm and standard deviation 0.12 cm. If 20% of the fish are longer than 13 cm, find the value of m.

24 A student takes 25 Mathematics tests over an entire year, with each test marked out of 10.

He inspects the results of these tests, and finds that

$$\sum_{r=1}^{25} x_r = 109 \quad \text{and} \quad \sum_{r=1}^{25} x_r{}^2 = 579,$$

where x_r refers to the score obtained in the rth selected test.

Find:

- **a** the mean score obtained in the student's tests over the year
- **b** the variance of the scores obtained during the year.

25 Consider the two lines

$$L_1: \frac{x-1}{2} = \frac{y-3}{3} = \frac{z-1}{2} \quad \text{and}$$

$$L_2: \frac{3-x}{4} = \frac{2y-3}{3} = \frac{z+1}{2}.$$

- **a** Write the vector equation of each line in the form $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{u}$.
- **b** Show that the two lines do not intersect, and state whether or not they are parallel.
- **c** Find, in the form $ax + by + cz = d$, an equation of a plane that is perpendicular to L_2 and which intersects L_1.
- **d** Find a vector that is perpendicular to both lines.
- **e** Hence, or otherwise, find the distance between L_1 and L_2.

Exam Set 7 NO CALCULATORS

1

- **a** Find all non-real complex numbers z such that $z^2 = 2\bar{z}$.
- **b** Given $z = x + 2i$, where $x \in \mathbb{R}$, for what values of x is $|z| < 2\,|z - 1 - i|$?

2 Solve for x: $\log_6(x+3) = 1 - \log_6(x-2)$.

3 Find the coordinates of the stationary point of the curve $y = xe^{-x^2}$.

4 The first term of an infinite geometric series exceeds the second term by 9. The sum of the series is 81. Find:

- **a** the common ratio of the series
- **b** the first term of the series.

5 Suppose $f(x) = \dfrac{x}{x-2}$, $x \neq 2$, and $g(x) = (f \circ f)(x)$. Find:

- **a** $g(x)$
- **b** $(g \circ g)(2)$.

6 A and B are acute angles such that $\sin A = \frac{4}{5}$ and $\sin B = \frac{8}{17}$.

Find the exact value of $\tan(A+B)$.

7 Find the exact values of x such that $0 \leqslant x \leqslant 2\pi$ and $\sqrt{2}\sin x = \tan x$.

8

- **a** Given $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, find $\mathbf{a} \times \mathbf{b}$.
- **b** Find a vector of length 5 units which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

9 The function $f(x) = \dfrac{ax+b}{x^2 - 5x + 7}$ has a turning point at (3, 5). Find the values of a and b.

10 A particle is moving along a straight line. After t seconds, its displacement from O is given by $s = A\cos 2t + B\sin 2t$ metres where $A, B \in \mathbb{R}$. The particle starts from rest at $s = 5$.

- **a** Show that the particle's acceleration is given by $a = -4s \text{ m s}^{-2}$.
- **b** Find A and B.
- **c** Find the velocity of the particle at the time when it first has a displacement of 3 metres.

11 Find the area enclosed by the curve $y^2 + 2y - 3x = 0$ and the line $2x - y - 1 = 0$.

12 Solve exactly for $x \in]-\infty, \infty[$, $\dfrac{5}{x+2} \geqslant \dfrac{2}{x+3}$.

13 Find the exact value of $\tan 75°$.

14

- **a** Show that the lines

$$\mathbf{r}_1 = \begin{pmatrix} 3+4t \\ 4+t \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} -1+12\lambda \\ 7+6\lambda \\ 5+3\lambda \end{pmatrix}$$

  intersect, and find the coordinates of the point of intersection.
- **b** Find an equation of the plane containing these 2 lines.

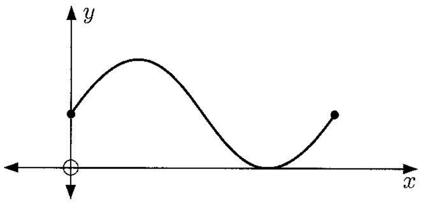
15 Use integration by parts to find $\int x \arctan x \, dx$.

16 If $\log(x^2y^3) = a$ and $\log\left(\dfrac{x}{y}\right) = b$, express $\log x$ and $\log y$ in terms of a and b.

17 Find the value of $\theta \in [0, \pi]$ if $2\cos\theta + 2\sec\theta = 5$.

18 The graph of $y = a\sin bx + c$ where $a, b, c \in \mathbb{R}$ and $0 \leqslant x \leqslant \dfrac{2\pi}{b}$ is drawn below.

On the same set of axes, sketch the graph of $y = -2a\sin\left(\frac{b}{2}x\right)$.



19 Find k if $y = Ae^{kt}$ and $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0$.

20 Find all points on the graph of $y^2 = 3 - xy$ where the tangents have gradient $-\frac{3}{4}$.

21 Given that events A and B are independent events with $\text{P}(A \cap B) = 0.3$ and $\text{P}(A \cap B') = 0.4$, find $\text{P}(A \cup B)$.

22 If $\sin A = -\frac{2}{3}$ and $0 < A < \frac{3\pi}{2}$, find the value of $\tan 2A$.

23 **a** Let $z = r(\cos\theta + i\sin\theta)$. By using $z = re^{i\theta}$, show that $z^n = r^n(\cos n\theta + i\sin n\theta)$.

b **i** Express $1 + i$ and $1 - i$ in the form $r\,\text{cis}\,\theta$ where r and θ are given as exact values and $r > 0$.

ii Hence prove that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right)$.

iii Prove that there is no integer n for which $(1+i)^n + (1-i)^n = 64$.

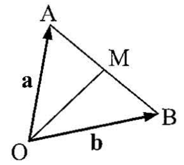
24 **a** Find the exact value of the the area of the region bounded by the graph of $y = \cos^2 x$, the x-axis, the y-axis, and $x = \frac{\pi}{2}$.

b Use integration by parts to prove that, for $n \in \mathbb{Z}$, $n \geqslant 3$

$$\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

c Hence find the volume of the solid generated by rotating the region in **a** through 2π about the x-axis.

25 Suppose **a** and **b** are the position vectors of points A and B, and that triangle OAB is equilateral. Let M be the midpoint of [AB].



a Express $\overrightarrow{\text{OM}}$ in terms of **a** and **b**.

b Use vectors to prove that $\overrightarrow{\text{OM}} \perp \overrightarrow{\text{AB}}$.

c Explain why $|\mathbf{b} - \mathbf{a}| = |\mathbf{a}|$, and use this result to prove that $\mathbf{b} \bullet (\mathbf{b} - 2\mathbf{a}) = 0$.

d Illustrate this result on a sketch of $\triangle$OAB.

Exam Set 8 CALCULATORS

1 On the first day of an exercise program, Paula jogs 500 metres. Each day she plans to jog 50 metres further than her distance on the previous day.

a How far will she jog on the 30th day?

b What would be the total distance Paula has jogged after 30 days?

2 Find the coefficients of $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ in the expansion of $(x + \dfrac{1}{x})^9$.

3 The weight W_t of radioactive uranium remaining after t years is given by the formula $W_t = W_0\,e^{-\frac{t}{5000}}$ grams, $t \geqslant 0$. Find:

a the *half-life* or time required for the weight to fall to 50% of its original value

b the time required for the weight to fall to 0.1% of its original value

c the percentage weight loss after 1000 years.

4 Three suppliers A, B, and C produce respectively 40%, 25%, and 35% of the total number of a certain component used by a washing machine manufacturer. The percentages of faulty components in each supplier's output are 5%, 3%, and 4% respectively.
Find the probability that a component selected at random is faulty.

5 Find the area of the region enclosed by the graphs of $y = e^{0.1x}$ and $y = 2\ln x$.

6 Consider the points A(3, 1, -2), B(1, 0, 4), and C(8, 3, 0).

a Show that $\overrightarrow{\text{AB}}$ is perpendicular to $\overrightarrow{\text{AC}}$.

b Find the size of angle ABC.

7 Visitors to an island have increased by 6% per annum each year since the year 2010.

a If 4000 people visited the island in 2010, how many would be expected to visit the island in 2020?

b For each of the years 2010 to 2020 inclusive, each visitor to the island is charged \$5. What is the total amount the visitors are expected to pay over these years?

8 **a** Find the domain of the function $f : x \mapsto \ln x + \ln(x+3) - \ln(x^2 - 9)$.

b Show that $f(x)$ can be written in the form $\ln\left(\dfrac{x}{x-3}\right)$.

c Find an expression for $f^{-1}(x)$.

9 Use the substitution $u = 1 - x$ to find $\int x^2\sqrt{1-x}\,dx$.

10 Find the real number a, for which $1 + ai$ is a zero of the quadratic polynomial $x^2 + ax + 5$.

11 Determine the quartic polynomial $f(x)$ which cuts the x-axis at -2 and 3, touches the x-axis at 1, and cuts the y-axis at -12. Give your answer in expanded form.

12 How many times must a pair of dice be rolled so there is a better than 70% chance of obtaining a roll where the same number appears on both dice?

13 In how many ways can nine different raffle tickets be divided between two students so that each student receives at least one raffle ticket?

14 Z is the standardised normal random variable with mean 0 and variance 1. Find a such that $\text{P}(|Z| \leqslant a) = 0.72$.

15 Find the equation of the tangent to the curve $x^2 + x\ln y - y = 3$ at the point $(2, 1)$.

16 Two circles with radii 5 cm and 8 cm respectively are such that their centres are 10 cm apart. Find the area common to the two circles.

17 The velocity of a particle at time t seconds is given by $v = e^t \cos 2t$ m s^{-1}.

a Write down an expression for the distance travelled by the particle in the first five seconds.

b Find the distance travelled by the particle in the first five seconds.

18 Let $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{s} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\mathbf{t} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, be the position vectors of the points R, S, and T respectively. Find the area of the triangle RST.

19 A continuous probability density function is described by:

$$f(x)=\begin{cases}2-5x, & \text{for } k \leqslant x \leqslant 0\\ 0, & \text{otherwise.}\end{cases}$$

a Find the value of k.

b Find the mean of the distribution.

20 A professional typist makes on average 2 errors per 500 words of typing. Find the probability that when typing a 1500 word essay, the typist will:

a make no more than 5 errors

b make no more than 5 errors given that at least one error has been made.

21 Consider points $A(4,\ 2,\ -1)$, $B(2,\ 1,\ 5)$, and $C(9,\ 4,\ 1)$.

a Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.

b Find an equation of the plane containing A, B, and C, and hence determine the distance from this plane to the point $(8,\ 1,\ 0)$.

c Find an equation of the line through A and B.

d Determine the distance from $D(8,\ 11,\ -5)$ to the line through A and B.

22 The function f is defined by $f(x)=e^{\sin^2 x}$, $0 \leqslant x \leqslant \pi$.

a Find an expression for $f'(x)$.

b Hence find the exact coordinates of the minimum and maximum turning points of the graph of $y=f(x)$.

c Find $f''(x)$.

d Find the equation of the tangent to the curve $y=f(x)$ at the point where $x=\frac{3\pi}{4}$.

e Find the exact value of the x-intercept of the tangent found in **d**.

f Find the area enclosed by the graphs $y=f(x)$ and $y=1$.

23 The random variable X has a normal distribution with mean 90. Given that $P(X<88)\approx 0.28925$, find the proportion of scores between 90 and 92.

24 **a** **i** Using mathematical induction, prove that

$$\sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4} \quad \text{for } n\in\mathbb{Z}^+.$$

ii Hence find $1^3+2^3+3^3+....+100^3$.

b Prove that $\sum_{r=1}^{n} r^3=\left(\sum_{r=1}^{n} r\right)^2$.

25 An alarm clock is used to wake a swimmer for early morning training. The probability that the alarm rings is $\frac{9}{10}$.

If the alarm rings, there is probability $\frac{5}{6}$ that the swimmer arrives for training, but if the alarm does not ring, the probability that the swimmer arrives for training is $\frac{2}{15}$.

a Find the probability that the swimmer arrives for early morning training on a given day.

b Find the probability that, on a randomly chosen morning on which the swimmer does not arrive for training, the alarm did not ring.

Exam Set 9 NO CALCULATORS

1 **a** Find integers a and b such that $(a+3i)(b-i)=13+i$.

b The complex number $z=5+ai$ where $a\in\mathbb{Z}$, is such that $|z+1+i|=2\,|z-2-i|$. Find the value of a.

2 In an arithmetic sequence of 30 terms, the sum of the even numbered terms exceeds the sum of the odd numbered terms by 8. Find:

a the common difference, d

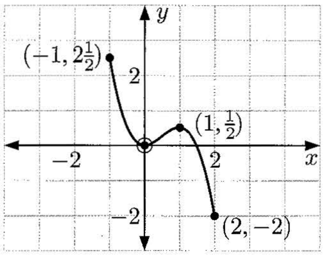
b an expression for S_n, the sum of the first n terms.

3 Find the area enclosed by the curve $y=x\sqrt{4-x^2}$ and the x-axis in the first quadrant.

4 Find the domain and range of $f\circ g$, where $f:x\mapsto\sqrt{x}$ and $g:x\mapsto 1-\sin x$.

5 Find all real values of x for which $2\,|x-3|\leqslant|x+7|$.

6 Alongside is the graph of $y=f(x)$.
It has local minimum and maximum points at $(0,\ 0)$ and $(1,\ \frac{1}{2})$ respectively.



a On the same diagram, draw the graph of $y=f(x-1)+\frac{3}{2}$.

b State the coordinates of the local minimum and maximum points of $y=f(x-1)+\frac{3}{2}$.

7 Simplify $\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}+\theta\right)\csc(\pi-2\theta)$.

8 Find $\int \sin^3 x\cos^3 x\,dx$.

9 **a** Find the coordinates of the points on the curve $3x^2+2xy-y^2=7$ where $x=-2$.

b Find the equation of the normal to the curve at one of these points.

10 Consider two vectors **p** and **q** such that $(\mathbf{p}+\mathbf{q})\bullet(\mathbf{p}+\mathbf{q})=25$ and that $|\mathbf{p}|^2+|\mathbf{q}|^2=25$.

a Find $|\mathbf{p}+\mathbf{q}|$.

b Prove that $\mathbf{p}\bullet\mathbf{q}=0$.

c Draw a diagram that represents the relative positions of **p**, **q**, and $\mathbf{p}+\mathbf{q}$.

11 Consider two complex numbers w and z such that $|w|=|z|$, $\arg(w)=\frac{\pi}{6}$, and $\arg(z)=\frac{\pi}{3}$. Find the exact values of:

a $\arg(wz)$

b $\arg(w+z)$.

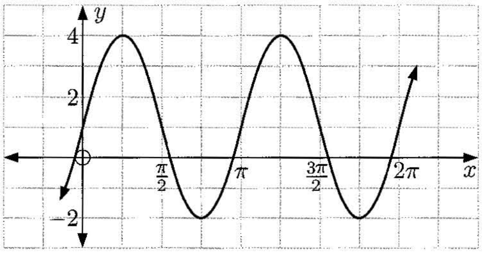
12 Find the domain and range of the function $f:x\mapsto|x-2|-4\,|x+1|$.

13 Find where the graphs of the functions $xy=2$ and $x^3+y^3=9$ meet.

14 The graph of the function $y=6x^2+px+q$ cuts the x-axis at 2 and $-\frac{1}{2}$.
Find the values of p and q.

15 If $\tan\beta=\frac{2}{3}$ and $\pi<\beta<\frac{3\pi}{2}$, find the exact value of $\tan\left(\frac{\beta}{2}\right)$.

16 The graph of $y = a\sin bx + c$, where a, b, and c are integers, is shown below.



Find the value of: **a** a **b** b **c** c

17 Find the x-coordinate of the point of inflection on the graph of $y = x^2 \ln\left(\frac{1}{x^2}\right)$, $x > 0$.

18 In triangle ABC, BC $= a$, AC $= b$, and AB $= c$. D is a point on the segment [BC] such that AD $= b$ and BD : DC $= 1 : 3$. $\widehat{ABC} = 60°$.

- **a** Use the cosine rule to find two different expressions for b^2.
- **b** Hence find an expression for a in terms of c.

19 If $f(x) = 3x^2 + 5x - 2$, find $f'(x)$ from first principles.

20 Use integration by parts to find $\int e^x \sin x \, dx$.

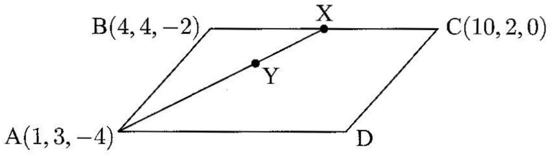
21 Consider the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -t \\ 1+t \\ 2t \end{pmatrix}$.

Find t such that:

- **a** $\mathbf{a}$ and $\mathbf{b}$ are perpendicular
- **b** $\mathbf{a}$ and $\mathbf{b}$ are parallel.

22 Find the exact values of x if $0 \leqslant x \leqslant \pi$ and $2\sin^2 x - \sin x - 2\sin x\cos x + \cos x = 0$.

23 In the figure, ABCD is a parallelogram. X is the midpoint of [BC], and Y is on [AX] such that AY : YX $= 2 : 1$.



- **a** Find the coordinates of D, X, and Y.
- **b** Prove that B, Y, and D are collinear.

24 **a** **i** Write $1 - i\sqrt{3}$ and $1 - i$ each in the form $r \operatorname{cis} \theta$.

ii Hence simplify $\dfrac{\left(1 - i\sqrt{3}\right)^{11}}{(1-i)^{18}}$, giving your answer in the form $x + iy$ where x and y are real numbers.

b Solve $z^5 = \sqrt{3} - i$, giving your answers in the form $r \operatorname{cis} \theta$.

25 Consider $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

- **a** Find $f'(x)$ and hence show that the graph of $y = f(x)$ has no stationary points.
- **b** Find $f''(x)$.
- **c** Find the exact value of $\displaystyle\int_0^{\ln 3} f(x)\, dx$, giving your answer in the form $\ln k$ where k is a rational number.

Exam Set 10 CALCULATORS

1 Each year, the population of seals on an island increases by 3%. In 2005 the population was 1250. Assuming the same rate of growth continues, during what year would the population be expected to exceed 2000 for the first time?

2 Find the angle between the plane $2x + 2y - z = 3$ and the line $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, $\mu \in \mathbb{R}$.

3 How many different arrangements, each consisting of 5 different digits, can be formed from 1, 2, 3, 4, 5, 6, and 7, if:

- **a** each arrangement begins and ends with an odd digit
- **b** odd and even digits alternate?

4 A and B are independent events with $\mathrm{P}(A) = 0.27$ and $\mathrm{P}(A \cup B) = 0.8$. Find $\mathrm{P}(B)$.

5 **a** Sketch the graph of the function $f(x) = \cos 2x - \sin 2x$, $0 \leqslant x \leqslant \pi$.

b Find the maximum value of $\cos 2x - \sin 2x$ on this interval.

6 Given $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$, and $\mathbf{a} \times \mathbf{b} = 12\mathbf{j} - 5\mathbf{k}$, find the possible values of $\mathbf{a} \bullet \mathbf{b}$.

7 A sample of 80 batteries was tested to see how long they last. The results are shown opposite.

Use mid-interval values to estimate:

- **a** the mean battery life
- **b** the standard deviation in battery life.

Time (t hours)	*Number of batteries*
$0 \leqslant t < 10$	3
$10 \leqslant t < 20$	6
$20 \leqslant t < 30$	9
$30 \leqslant t < 40$	8
$40 \leqslant t < 50$	12
$50 \leqslant t < 60$	14
$60 \leqslant t < 70$	11
$70 \leqslant t < 80$	8
$80 \leqslant t < 90$	7
$90 \leqslant t < 100$	2
Total	80

8 An employer randomly selects five new employees from twelve applicants consisting of five men and seven women.

- **a** Find the probability that no men are selected.
- **b** Find the probability that 3 women and 2 men are selected.

9 O and B are fixed points such that OB $= 8$ cm. A is a point on a circle with centre O and radius 5 cm. A lies initially on [OB], and rotates in a counter-clockwise direction about O, completing one rotation every 2 seconds. Find the rate of change in the area of triangle AOB when A is moving away from [OB] and $\widehat{AOB} = 30°$.

10 Find: **a** $\int \tan^5 x \, dx$ **b** $\displaystyle\int \frac{1}{x^2} \ln x \, dx$

11 The probability distribution of a discrete random variable X is given by $\mathrm{P}(X = x) = a\left(\frac{1}{7}\right)^x$, for $x = 0, 1, 2, 3, \ldots$.

Find the value of a.

12 A function is defined by $f(x) = x^3 + 3x^2 + bx + 4$, where b is a constant.

- **a** Find the coordinates of the point of inflection of this function.
- **b** For what values of b would the point of inflection found in **a** be a stationary point of the function?
- **c** For what values of b would the function have no stationary points?

13 Two functions f and g are defined by
$f : x \mapsto 2x - 3$ and $g : x \mapsto 3(2 - x)$.
Find: **a** $(f \circ g)(-4)$ **b** $f^{-1}(2)$

14 20% of students do *not* obtain a score of 35 or more in their IB aggregate. Find the probability that out of 9 randomly selected IB students, exactly 6 of them will obtain a score of 35 or more in their IB aggregate.

15 A and B are acute angles such that $\sin A = \frac{3}{5}$ and $\tan(A + B) = -\frac{63}{16}$. Show that $\cos B = \frac{5}{13}$.

16 Prove, using mathematical induction, that

$$\sum_{r=1}^{n}(r^2 + r) = \frac{n(n+1)(n+2)}{3} \quad \text{for } n \in \mathbb{Z}^+.$$

17 Consider the two planes P and Q, with equations

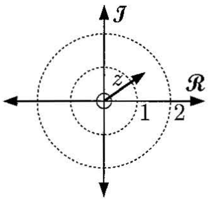
$$P : \mathbf{r} \bullet \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} + 8 = 0 \qquad Q : \mathbf{r} \bullet \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = 0.$$

Find an equation of a third plane R which is perpendicular to both P and Q, and contains the point $(2, 2, 3)$.

18 A publisher finds that in the draft of a new 750 page book, the probability of at least one typing error per page is 0.01. Suppose X is the Poisson random variable representing the number of errors per page. Find the expected number of:

a errors per page **b** pages with exactly one error.

19 The complex number z is such that $1 < |z| < 2$ and $0 < \arg(z) < \frac{\pi}{4}$.
The position of z is shown on the Argand diagram alongside.
Mark on the diagram the positions of:



a $-z$ **b** $\bar{z}$ **c** iz **d** z^2 **e** $\sqrt{z}$

20 The mean test score for a mathematics class was 64, and the standard deviation was 8.352. Assuming that the test scores were normally distributed, find the proportion of students scoring at least 80 for the test.

21 P is a point in the first quadrant on the graph of $y = 10 - xe^x$. A and B are points on the x and y axes respectively, such that OAPB is a rectangle. Find the maximum possible area of the rectangle.

22 Find the area enclosed by the curves

$$y = x \arccos\left(\frac{x-1}{2}\right) + 1 \quad \text{and} \quad y = x^2 - 1.$$

23 The function f is defined by $f(x) = \dfrac{x^2 + 1}{e^x}$.

a Determine the behaviour of $f(x)$ as $x \to \infty$.

b Find $f'(x)$, fully simplifying your answer.

c Show that $\left(1, \dfrac{2}{e}\right)$ is a stationary point, and state its nature.

d Find an expression for $f''(x)$, fully simplifying your answer.

e Hence find the exact coordinates of the non-stationary point of inflection on the graph of f.

f Sketch the graph of $y = f(x)$ for $0 \leqslant x \leqslant 5$. Clearly show the two points found in **c** and **e**, and the exact coordinates of the endpoints of the graph.

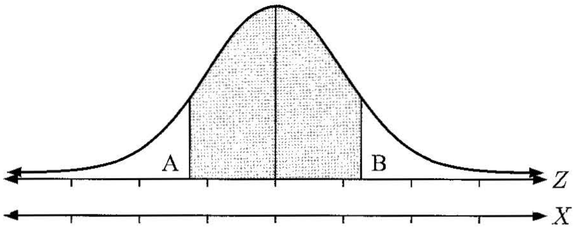
24 The depth of water in a shipping channel at the entrance of a harbour is given by $x = 6 + 2\cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right)$ metres, where t is the number of hours after midday on Sept 1.

a At what time will the first high tide occur?

b How much time elapses between consecutive high tides?

c A ship in the harbour is ready to leave at 3.00 pm on September 2. If it needs a depth of at least 5.5 metres, what is the earliest time it could safely exit the harbour?

25 The following normal distribution graph models the age at death X for people in a developing country. The mean is 43 years and the standard deviation is 7 years.



a Complete a Z-scale and an X-scale on the given number lines.

b The unshaded region A corresponds to 9% of the population. What is the greatest age at death of any person in the smallest 9% of the population?

c The unshaded region B corresponds to 11% of the population. Find the lowest age at death of any person in the top 11% of the population.

d Find the area of the shaded region.

Exam Set 11 NO CALCULATORS

1 **a** Write -8 in the form $r \operatorname{cis} \theta$.

b Find the cube roots of -8 in the form $a + bi$ where $a, b \in \mathbb{R}$.

2 The function f is defined by $f : x \mapsto \sqrt{5 - 2x}$, $x \leqslant \frac{5}{2}$.
Find $f^{-1}(5)$.

3 Suppose $f(x) = x + 2$. Find the equation of the function F obtained by stretching f vertically by a factor of 2 and horizontally by a factor of $\frac{1}{2}$, then translating the result by $\frac{1}{2}$ horizontally and -3 vertically.

4 Let $f(x) = 2x^3 - 9x^2 + 30x - 13$.

a Find $f(\frac{1}{2})$.

b Factorise $f(x)$ completely into linear factors.

5 Find the exact value of $\tan A$ if $\tan\left(A + \frac{\pi}{4}\right) = 3$.

6 Consider the equation $2^{x+1} = 5^{x-1}$.
Find the exact value of x, giving your answer in terms of $\log_2 5$.

7 Sketch the graph of $\sin(x + y) = 0$ for $-2\pi \leqslant x \leqslant 2\pi$ and $0 \leqslant y \leqslant 4\pi$.

8 Find a if $\displaystyle\int_{-a}^{a} (3x^2 - 8x + 2)\, dx = 12a$ and $a > 0$.

9 The function f is defined by $f(x) = 4xe^{-x}$ for $x \geqslant 0$.
For the graph $y = f(x)$, find the coordinates of:

a the turning point **b** the point of inflection.

10 Consider the geometric series $1 - \tan^2 x + \tan^4 x - .....$

- **a** Determine the values of x, where $0 \leqslant x \leqslant 2\pi$, for which the sum of the series converges.
- **b** For the values of x found in **a**, determine the sum of the series.
- **c** For what values of x in the interval $[0, 2\pi]$ does the sum of the series equal $\frac{1}{2}$?

11 State the range of values that k may take if the system of equations $\begin{cases} x - y - z = 1 \\ 2x + y + z = 2 \\ 4x - y - z = k \end{cases}$ has:

a an infinite number of solutions **b** no solution.

12 Use mathematical induction to prove that, for all $n \in \mathbb{Z}^+$, $2^{n-1} \leqslant n!$

13 Suppose $P(N) = \frac{3}{5}$, $P(M \mid N) = \frac{3}{7}$, and $P(M \mid N') = \frac{1}{6}$.
Find: **a** $P(N')$ **b** $P(M' \cap N')$.

14 Solve for x given $0 \leqslant x \leqslant 2\pi$:

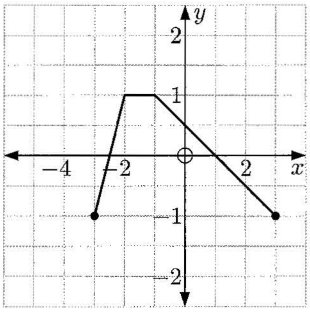
a $\sin 2x + \cos 2x = 1$ **b** $\cos 2x = -\cos x$

15 Use the substitution $u = 1 - x$ to find $\displaystyle\int \frac{x^2}{(1-x)^3}\,dx$.

16 **a** Differentiate $\dfrac{x}{\sqrt{x-6}}$, simplifying your answer.

- **b** Hence find the x-coordinate of the minimum turning point of the function $y = \dfrac{x}{\sqrt{x-6}}$, $x > 6$.

17 The graph of $f(x)$ is given.



- **a** Draw the graph of $y = |f(x)|$, on the same set of axes.
- **b** Find the y-intercept of $y = \dfrac{1}{f(x)}$.
- **c** Draw the graph of $y = \dfrac{1}{f(x)}$ on the same set of axes.

18 **a** For what values of b does the infinite series $a + \dfrac{a}{b^2} + \dfrac{a}{b^4} + ...$ converge?

- **b** For the values of b found in **a**, find an expression for the sum of the series.
- **c** By making suitable substitutions for a and b, express $0.\overline{32}$ in the form $\dfrac{p}{q}$, $p, q \in \mathbb{Z}$.

19 The area bounded by the curve $y = \dfrac{x-3}{x}$, the x-axis, and the line $x = 6$, is rotated through 2π about the x-axis. Find the exact value of the volume of the solid generated.

20 Simplify $\dfrac{\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^5 \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^7}$
giving your answer in the form $a + bi$ where $a, b \in \mathbb{R}$.

21 Let the vectors **a**, **b**, and **c** be
$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

- **a** Show that $\mathbf{b} \times \mathbf{c} = -4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.
- **b** Verify for the given vectors that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$.

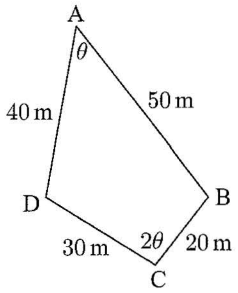
22 **a** Write down the solutions of $w^6 = 64$, giving your answers in the form $x + iy$.

- **b** Hence, determine the solutions of the equation $(z+1)^6 = 64(z-2)^6$.

23 Given the sequence defined by $u_{n+2} = u_n + u_{n+1}$ where $u_1 = u_2 = 1$ and $n \in \mathbb{Z}^+$, use mathematical induction to prove that

$$u_n = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

24 Field ABCD is a quadrilateral with sides $AB = 50$ m, $BC = 20$ m, $CD = 30$ m, and $AD = 40$ m.
$B\widehat{A}D = \theta$ and $B\widehat{C}D = 2\theta$.



- **a** Find the value of $\cos\theta$.
- **b** Hence find the area of the field.

25 The function f is defined by $f(x) = \dfrac{\sin x}{\cos x + \sqrt{2}}$, for $0 < x < 2\pi$.

- **a** **i** Find $f'(x)$.
  - **ii** Find the exact coordinates of the turning points of $y = f(x)$.
- **b** **i** Find $f''(x)$.
  - **ii** Find the exact coordinates of the points of inflection of the graph of $y = f(x)$.
- **c** Sketch the graph of $y = f(x)$ for $0 < x < 2\pi$.
- **d** Find the exact value of the area enclosed by the graph of $y = f(x)$, the x-axis, $x = \frac{\pi}{4}$, and $x = \frac{3\pi}{4}$.

Exam Set 12 CALCULATORS

1 Find the coefficient of x^7 in the expansion of $(1+2x)(2-x)^8$.

2 On Peter's first birthday, his grandmother gave him \$100. For each subsequent birthday, she gives him 10% more than on his previous birthday.

- **a** Write an expression for how much Peter receives on his nth birthday.
- **b** Write an expression for the total amount Peter has received for the first n birthdays.
- **c** On which birthday does the total amount received by Peter first exceed \$4000?

3 Using the digits 2, 3, 4, 5, 6, 7, how many 3 digit numbers can be formed if:

- **a** repetition of digits is permitted
- **b** digits cannot be repeated
- **c** digits cannot be repeated and the numbers formed are greater than 400 and even
- **d** the digits in each number are in ascending order from left to right?

4 Find the exact value of u if $\displaystyle\int_0^2 \frac{1}{1+ux}\,dx = \frac{1}{u}$.

5 The line $x+1=\dfrac{y-1}{2}=\dfrac{z-1}{-2}$ and the point $(1,\ 3,\ -2)$ lie on the plane P.

- **a** Find an equation for P.
- **b** Calculate the shortest distance from the plane to the origin.

6 A bag contains 4 red and 9 blue balls. If two balls are drawn at random without replacement, what is the probability that one of them is red and the other is blue?

7 Sketch the graph of $y=x\arcsin\left(\dfrac{x}{3}-0.5\right)$ for $-1\leqslant x\leqslant 3$.
On your graph, indicate the coordinates of the endpoints, axes intercepts, and turning point.

8 The diameter of the circular top of an inverted conical container is 20 cm. The height of the container is 20 cm, and the container is being filled with water at a constant rate of 30 $\text{cm}^3\text{s}\ ^{1}$. Find the rate at which the depth of water is increasing, when the depth is 15 cm.

9 A player rolls a biased tetrahedral die. The probability of each possible score is shown below.

Score	1	2	3	4
Probability	$\frac{2}{7}$	$\frac{1}{3}$	x	$\frac{2}{21}$

Find the probability of a total score of six after two rolls.

10 Consider the vectors $\mathbf{u}=\begin{pmatrix}2\\2\\3\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$, and $\mathbf{w}=\mathbf{i}+(2-\lambda)\mathbf{j}+(\lambda+1)\mathbf{k}$. Find the parameter λ such that the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar.

11 The first term of an arithmetic sequence is 285 and the tenth term is 213.

- **a** Find, in terms of n, an expression for the nth term, u_n.
- **b** How many terms of the sequence are positive?

12 Find n if $3\binom{n}{2}=\binom{n}{3}$.

13 Customers arrive at random at a grocery store at the rate of 12 per hour. Find the probability that 5 customers will arrive at the store in a half-hour period.

14 In triangle ABC, AB $=5$ cm, BC $=7$ cm, $\text{A}\widehat{\text{C}}\text{B}=\theta$, and $\text{B}\widehat{\text{A}}\text{C}=2\theta$. Find θ.

15 In a certain university, 58% of the students are male. From the results of a survey, 73% of the female students and 48% of the male students do not own a car.

- **a** Find the probability that a student selected at random does not own a car.
- **b** If a student is selected at random and does not own a car, what is the probability that the student is male?

16 Use mathematical induction to prove that, for all $n\in\mathbb{Z}^+$, $\dfrac{d^n}{dx^n}(xe^x)=(x+n)e^x$.

17 Find the value of a where $a>0$, if:

- **a** $\displaystyle\int_0^a x\sqrt{1-x^2}\,dx=0.2$
- **b** $\displaystyle\int_0^a \frac{x}{x^2+1}\,dx=1.$

18 Find the area of the parallelogram determined by the vectors $\mathbf{a}=3\mathbf{i}-4\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}+\mathbf{j}-5\mathbf{k}$.

19 The velocity of a particle moving in a straight line is given by $v=4t^3-9t^2+2\ \text{m s}^{-1}$, where t is the time in seconds, $t\geqslant 0$.

- **a** Find an expression for a, the acceleration of the particle after t seconds.
- **b** Find the minimum velocity of the particle.
- **c** Find an expression for s, the displacement from O of the particle at time t seconds, given that when $t=0$, $s=-6$ m.
- **d** Find the time when the particle first passes through O.
- **e** Find the total distance travelled in the first 5 seconds.

20 The scores in a Biology examination are distributed normally with mean 56% and standard deviation 30.512%.

- **a** Students needed to get 72% to get a "6" or better. What percentage of students gained a "6" or better?
- **b** If the pass mark was 40%, what percentage of students taking the Biology exam actually passed?
- **c** Micah obtained a score of 94%. If only the top 10% of students got a "7", what grade did Micah receive?
- **d** Micah received a score of 87% for English, a subject with marks normally distributed with mean 63% and standard deviation 18.31%.
  - **i** Determine whether Micah performed better in Biology or English.
  - **ii** What grade would he get in English if the same percentage of students got "7"s and "6"s in English as in Biology?

21 Let $f(x)=x^n,\ n\in\mathbb{Z}^+$

- **a** Express $f(x+h)$ in expanded form in decreasing powers of x, showing the first 3 and last 3 terms of the expansion.
- **b** Hence find $f'(x)$ from first principles.

22 Find the exact value of the volume of the solid formed by rotating $y=\sin x$, $0\leqslant x\leqslant\pi$, through 2π about the x-axis.

23 The functions f and g are defined by:
$f(x)=2x\sin x$ and $g(x)=x$ for $0\leqslant x\leqslant 2\pi$.

- **a** Find the exact coordinates of the points of intersection of the graphs $y=f(x)$ and $y=g(x)$.
- **b** Write down an equation which would give the x-coordinates of the stationary points of $y=f(x)$.
- **c** Find the coordinates of the stationary points of $y=f(x)$.
- **d** Sketch the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes.
- **e** Use integration by parts to find $\int 2x\sin x\,dx$.
- **f** Find the exact area enclosed by the graphs of $y=f(x)$ and $y=g(x)$.

24 An advanced mathematics class consists of 7 girls and 9 boys.

- **a** How many different committees of 7 students can be chosen from this class?
- **b** How many such committees can be chosen if class members Haakon (a male) and Josefine (a female) cannot both be on the committee?
- **c** How many committees of 7 students can be chosen if there must be more boys than girls on the committee?

25 Scooter was born on the 6th of January 2001 in Adelaide. The day was an unusually hot day of 40°C.

- **a** The daily maximum temperature in Adelaide in January is normally distributed with mean 33°C and standard deviation 3.5°C. Find the proportion of January days in Adelaide that will have a daily maximum temperature of more than 40°C.
- **b** While on holiday in Prague, Scooter met Pokey who was born on the same day (January 6th 2001) as he was. After some conversation they discovered that Pokey was born in Prague on a day when the minimum temperature was −12°C.

  Given that the daily minimum temperature in Prague in January is normally distributed with $\mu = -3.2$°C and $\sigma = 4.9$°C, calculate the proportion of January days in Prague that will have a daily minimum temperature of less than −12°C.
- **c** Use your knowledge of the normal distribution to explain which city experienced the more extreme temperature on the day when Scooter and Pokey were born.
- **d** How cold would it have to be in January in Prague for the weather to be comparable with Adelaide's hottest January day on record, which was 46.4°C?

TRIAL EXAMINATION 1

NO CALCULATOR **120 marks / 3 hours**

SECTION A (56 marks)

1 A discrete random variable X has the probability distribution:

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{6}$	$2k$	$\frac{1}{5}k$	$\frac{1}{3}$	$\frac{2}{5}k$

- **a** Find the value of k. (2 marks)
- **b** Calculate $P(0 < X < 4)$. (2 marks)
- **c** Find $E(X + 1)$. (2 marks)

2 $5x^2 + 2x - 3 = 0$ has roots α and β.

- **a** Without solving the equation directly, find the values of $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\frac{1}{\alpha\beta}$. (3 marks)
- **b** Hence find *all* quadratic equations which have roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. (2 marks)

3 In triangle ABC, $A\hat{B}C = 2\theta$, $A\hat{C}B = \theta$, AB = 3 cm, and AC = 5 cm.

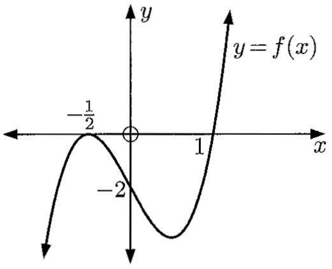
- **a** Show that $\cos\theta = \frac{5}{6}$. (3 marks)
- **b** Find $\sin(A\hat{B}C)$. (3 marks)

4 Find all values of x for which $\left|\frac{3x-1}{x+2}\right| > 1$. (6 marks)

5 The results of matches played by the local football team seem to depend on the temperature during the game. When it is hot, the probability that the local team wins is $\frac{3}{8}$, and when it is cold, the probability is $\frac{9}{16}$. During the season, 40% of the games are played in hot conditions.

- **a** Display this information on a suitable diagram. (1 mark)
- **b** On one particular day, the team won. Find the probability that the weather that day was hot. (4 marks)

6 Consider the illustrated graph of $y = f(x)$ where $f(x) = ax^3 + bx^2 + cx + d$.



- **a** Find the values of a, b, c, and d. (4 marks)
- **b** Find the y-intercept for the graph of $g(x) = 2f(x - 1)$. (1 mark)
- **c** Find the coordinates of the point of inflection of $y = g(x)$. (3 marks)

7 The radius of the base of a cylindrical can is r cm, and its height is $2r$ cm. Find the rate at which the surface area of the cylinder is increasing when the radius is 5 cm and the volume is increasing at 5π cm^3 s^{-1}. (7 marks)

8 Use the substitution $x = \sec\theta$, to find $\int \frac{\sqrt{x^2-1}}{2x}\,dx$ in simplest form. (6 marks)

9 If $y = xe^{-x}$, use mathematical induction to prove that $\frac{d^ny}{dx^n} = \frac{(-1)^{n+1}(n-x)}{e^x}$ for all $n \in \mathbb{Z}^+$. (7 marks)

SECTION B (64 marks)

10

- **a** On an Argand diagram, show vector representations for $z = r\operatorname{cis}\theta$ and $z + 1$, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. (3 marks)
- **b** Show that $|z + 1| = \sqrt{r^2 + 2r\cos\theta + 1}$. (3 marks)
- **c** If $\arg(z + 1) = \alpha$, show that
  $$\cos\alpha = \frac{1 + r\cos\theta}{\sqrt{r^2 + 2r\cos\theta + 1}}.$$ (4 marks)
- **d** If $|z| = 1$, use **c** to show that $\alpha = \frac{\theta}{2}$. (5 marks)
- **e** Give geometrical evidence to support the result in **d**. (1 mark)

11 Consider the function $f(x) = \frac{k\ln x}{x}$, $k > 0$, $x > 0$.

- **a** Find the coordinates of the stationary point on the graph of $y = f(x)$. (3 marks)
- **b** Find the coordinates of the point of inflection on the graph of $y = f(x)$. (3 marks)
- **c** Find k such that $\int_1^{e^2} \frac{k\ln x}{x}\,dx = 10$. (2 marks)
- **d** Use integration by parts to show that
  $$\int \frac{(\ln x)^2}{x^2}\,dx = -\frac{1}{x}(\ln x)^2 - \frac{2}{x}\ln x - \frac{2}{x} + c$$
  (5 marks)
- **e** Hence, find the exact value of the volume of the solid formed by rotating the region enclosed by the x-axis, $y = f(x)$ where $k = 1$, and the line $x = e^2$, through 2π about the x-axis. (3 marks)

12 A plane has equation $2x - y + z = 15$ and a line has vector equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -a \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ a \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

The line is perpendicular to the plane.

- **a** Find:
  - **i** the value of a (3 marks)
  - **ii** the coordinates of the point where the line meets the plane. (3 marks)
- **b** Point $P(3, -1, 2)$ does not lie on the plane. Find:
  - **i** $\overrightarrow{PA}$ if A is a general point on the given line (2 marks)
  - **ii** λ such that $\overrightarrow{PA}$ is perpendicular to the line (2 marks)
  - **iii** the coordinates of the foot of the perpendicular from P to the line (1 mark)
  - **iv** the shortest distance from P to the line. (2 marks)
- **c** $2x - y + z = 15$ and $x + 3y - 2z = 8$ are two planes. Find the equation of a third plane which is perpendicular to both these planes and which cuts the x-axis at -5. (3 marks)

13 Consider $f(x) = \dfrac{ae^{-x}}{b - ae^{-x}}$ where $a > 0$, $b > 0$.

- **a** Show that $f'(x) = \dfrac{-abe^{-x}}{(b - ae^{-x})^2}$. (3 marks)
- **b** Deduce that $f''(x)$ is never zero. (5 marks)
- **c** Find the equations of the asymptotes of f. (2 marks)
- **d** Draw a sign diagram for $f'(x)$. (1 mark)
- **e** Suppose $a = 3$ and $b = 1$.
  - **i** Draw a fully labelled graph of f. (2 marks)
  - **ii** Find the area of the region enclosed by f, the axes, and the vertical line $x = \ln 2$. (3 marks)

CALCULATOR 120 marks / 3 hours

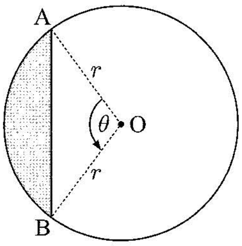
SECTION A (54 marks)

1 $\prod_{n=1}^{10} (nx - 1) = (x-1)(2x-1)(3x-1) \ldots . (10x - 1)$.

- **a** Find k given that

  $$\prod_{n=1}^{10} (nx - 1) = k(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3}) \ldots . (x - \tfrac{1}{10}).$$
  (2 marks)
- **b** Hence find the integer coefficient of x^9 in the expansion of $(x-1)(2x-1)(3x-1) \ldots . (10x-1)$. (4 marks)

2 [AB] is a chord of a circle with centre O and radius r. $\widehat{AOB} = \theta$ radians.



- **a** Prove that the shaded segment has area $A = \frac{1}{2}r^2(\theta - \sin\theta)$. (2 marks)
- **b** Chord [AB] divides the circle into two segments such that the ratio of areas shaded : unshaded = 1 : 3. Find θ in degrees, to one decimal place. (6 marks)

3 Consider $f(x) = \begin{cases} k(x^2 + 3), & \text{for } 0 \leqslant x \leqslant 2 \\ 0, & \text{elsewhere.} \end{cases}$

- **a** Show that for $f(x)$ to be a well defined probability density function, $k = \frac{3}{26}$. (2 marks)
- **b** Find $P(X \geqslant 0.6)$. (1 mark)
- **c** Find the median of the distribution of X. (3 marks)

4 Suppose X is a binomial random variable where the number of trials is 12 and the probability of success at any trial is p. Find the possible values of p, given that $P(X = 3) = \frac{1}{10}$. (5 marks)

5 $2\cos 2\theta + 1 = 5\sin\theta$ where $\frac{\pi}{2} < \theta < \pi$.

- **a** Show that $4\sin^2\theta + 5\sin\theta - 3 = 0$. (2 marks)
- **b** Hence find θ, correct to 3 decimal places. (3 marks)

6 The random variable X is normally distributed with $P(X \geqslant 20) \approx 0.386$ and $P(X \geqslant 25) \approx 0.183$. Find the mean and standard deviation of X. (7 marks)

7 If $\log_a 3 = 7$, find:

- **a** the exact value of $\log_a 27$ (2 marks)
- **b** the exact value of $\log_{\sqrt{a}} 3$ (2 marks)
- **c** the value of a, to 3 significant figures. (2 marks)

8
- **a** Prove that $f(x) = x^2 \sin^3 x$ is an odd function. (2 marks)
- **b** Sketch the graph of $y = f(x)$ for $-\pi \leqslant x \leqslant \pi$. (2 marks)
- **c** Find $\int_{-a}^{a} x^2 \sin^3 x \, dx$. Give reasons for your answer. (2 marks)

9 15 mature scallops have shell widths (to the nearest mm) of:

70 72 80 74 74 67 71 78
72 68 76 72 70 75 71

- **a** Find the:
  - **i** mean **ii** variance. (2 marks)
- **b** Assuming the widths of scallops is normally distributed, with the mean and variance found in **a**, find the probability that a randomly chosen scallop will have width between 70 mm and 80 mm. (2 marks)
- **c** Would your answer to **b** be reliable? Explain your answer. (1 mark)

SECTION B (66 marks)

10
- **a** By making the substitution $u = \sqrt{x-1}$, show that:

  $$\int \frac{\sqrt{x-1}}{x} = 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + c.$$
  (4 marks)
- **b** Does $\int_{-1}^{1} \frac{\sqrt{x-1}}{x} \, dx$ exist? Explain your answer. (2 marks)
- **c** Find the exact value of $\int_{1}^{2} \frac{\sqrt{x-1}}{x} \, dx$. (2 marks)
- **d**
  - **i** Sketch the graph of $y = \dfrac{\sqrt{x-1}}{x}$. (2 marks)
  - **ii** Find the coordinates of the local maximum. (2 marks)

e **i** Draw the solid obtained by rotating the graph of $y = f(x)$ from $x = 1$ to $x = 10$ through 2π about the x-axis. (2 marks)

ii Find the exact volume of the solid formed. (3 marks)

11 Vectors **a** and **b** are the position vectors of points $A(-4, 12, 8)$ and $B(4, 8, 0)$ respectively.

a Find M, the point with position vector $\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$. (2 marks)

b State the ratio in which M divides [AB]. (2 marks)

c Find the point on (AB) that is three units from A in the direction of B. (4 marks)

d Find the equation of the plane containing A, B, and O. (4 marks)

e Find the size of the angle ABE, where E is point (0, 0, 1). (4 marks)

12 **a** Use $z = r \operatorname{cis} \theta = re^{i\theta}$ to prove De Moivre's theorem. (2 marks)

b Hence show that if $z = r \operatorname{cis} \theta$, then $\left|z^{-2}\right| = \dfrac{1}{r^2}$ and $\arg(z^{-2}) = -2\theta$. (2 marks)

c Suppose $z = \sqrt{3} - i$ and $w = 2 + 2i$.

i Write each of z and w in polar form. (4 marks)

ii Hence write $\dfrac{z^7}{w^4}$ in terms of z only. (3 marks)

d By completing the following argument, find a formula for $\cos 3\theta$ in terms of $\cos\theta$ only, and also a formula for $\sin 3\theta$ in terms of $\sin\theta$ only.

$$\begin{aligned} &\cos 3\theta + i \sin 3\theta \\ &= \operatorname{cis} 3\theta \\ &= [\operatorname{cis} \theta]^3 \qquad \{\text{De Moivre}\} \\ &= [\cos\theta + i\sin\theta]^3 \\ &\quad\vdots \end{aligned}$$

(3 marks)

e Hence find the exact solutions of the equation $3x - 4x^3 = \frac{1}{2}$. (3 marks)

13 Pam has a coin with '1' written on one side and '2' on the other. She also has a six-sided die on which one face has a '1', two faces have a '2', and three faces have a '3'.
When Pam tosses the coin and rolls the die, X is the random variable for the *sum* of the numbers on the coin and the die.

a Determine the probability distribution of X. (4 marks)

b Find: **i** $P(X = 4)$ **ii** $P(X \leqslant 3)$ (2 marks)

c Sam has his own coin and die marked the same as Pam's. Suppose he throws his objects at the same time.

i Find the probability that one of their sums is 3 and the other is 4. (2 marks)

ii Now suppose that Y is the random variable for the *least* of the four results when Pam and Sam throw their coins and dice simultaneously. Find:

(1) $P(Y = 3)$ **(2)** $P(Y = 2)$ **(3)** $P(Y = 1)$
(4) $E(Y)$ **(5)** $\text{Var}(Y)$

(8 marks)

TRIAL EXAMINATION 2

NO CALCULATOR **120 marks / 3 hours**

SECTION A **(63 marks)**

1 **a** Find the derivative of $x\cos x$, and hence find $\int x \sin x \, dx$. (2 marks)

b $f(x) = \begin{cases} k\sin x, & \text{if } 0 \leqslant x \leqslant \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}$

is a probability density function for the random variable X. Find:

i the value of k (2 marks)

ii the mean value of the random variable X. (2 marks)

2 The sum of the first eight terms of an arithmetic sequence is -4, and the sum of the next eight terms is 188.

a Find an expression for u_n, the nth term of the sequence. (5 marks)

b Hence find the number of terms needed in the corresponding series for the sum to equal 25. (2 marks)

3 **a** Expand and simplify $\left(2x - \dfrac{1}{x}\right)^4$. (3 marks)

b Check your expansion in **a** by substituting $x = 1$. (1 mark)

c Find $(2\sqrt{3} - \frac{1}{\sqrt{3}})^4$, giving your answer in simplest form. (2 marks)

4 Consider the two lines with equations:

$$L_1: \quad x = 3\lambda + 4, \quad y = \lambda + 4, \quad z = 2\lambda - 1,$$

$$L_2: \quad x + 1 = \frac{y-k}{2} = \frac{z-1}{-2}.$$

a Find the value of k for which these lines are coplanar, and hence determine the point of intersection of the two lines. (3 marks)

b Find a vector **p** which is perpendicular to both L_1 and L_2. (2 marks)

c Show that an equation of the plane with normal vector **p** and which contains L_1, is $-6x + 8y + 5z = 3$. (2 marks)

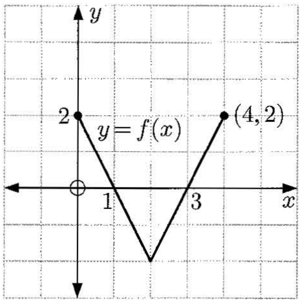
5 **a** Show that the antiderivative of $\cos^2\theta \sin\theta$ is $-\frac{1}{3}\cos^3\theta$. (2 marks)

b By making the substitution $x = \sin\theta$, find

$$\int \frac{x^3}{\sqrt{1-x^2}} \, dx.$$

(6 marks)

6 Consider the graph of $y = f(x)$ shown.



a Draw the graph of $y = |f(x)|$ on the same set of axes as shown. (1 mark)

b Find the y-intercept of $y = \dfrac{1}{f(x)}$. (1 mark)

c Draw the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. (2 marks)

d For the given function $f(x)$, find x such that $[f(x)]^2 = 1$. (2 marks)

7 Suppose X is a Poisson random variable with parameter m, and $m < 2$.

a The mean of $\text{P}(X = 0)$ and $\text{P}(X = 2)$ is $\text{P}(X = 1)$. Find m. (6 marks)

b Find the exact value of $\text{P}(X \leqslant 1)$. (2 marks)

8 **a** Simplify $(1 + \text{cis}\,\alpha)(1 - \text{cis}\,\alpha)$ and hence write $1 + \text{cis}\,\alpha$ in terms of α. (2 marks)

b Show that

$$1 + \text{cis}\,\alpha + \text{cis}\,2\alpha + \text{cis}\,3\alpha + ... + \text{cis}\,n\alpha$$
$$= \frac{1 - \text{cis}\,n\alpha}{1 - \text{cis}\,\alpha} \quad \text{provided } \text{cis}\,\alpha \neq 1 \text{ and } n \in \mathbb{Z}^+.$$

(3 marks)

c Hence simplify:

$1 + \text{cis}\left(\frac{\pi}{11}\right) + \text{cis}\left(\frac{2\pi}{11}\right) + ... + \text{cis}\left(\frac{22\pi}{11}\right)$. (2 marks)

9 Consider the vectors **a**, **b**, and **c** where $\mathbf{c} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$.

a If θ is the angle between **a** and **c**, show that

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b} + |\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}.$$

(3 marks)

b If ϕ is the angle between **b** and **c**, find $\cos\phi$. (2 marks)

c Explain why $\theta = \phi$. (1 mark)

d Given the vectors $\mathbf{a} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ find *all* vectors which bisect the angle between **a** and **b**. (2 marks)

SECTION B (57 marks)

10 **a** Prove that $\sin A \sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$ for all values of A and B. (2 marks)

b Hence, prove that:

- **i** $\sin 3x \sin x + \sin x \sin x = \sin^2 2x$ (2 marks)
- **ii** $\sin 5x \sin x + \sin 3x \sin x + \sin x \sin x = \sin^2 3x$ (3 marks)

c Hence, predict the sum of:

- **i** $\sin 3x + \sin x$
- **ii** $\sin 5x + \sin 3x + \sin x$
- **iii** $\sin 9x + \sin 7x + \sin 5x + \sin 3x + \sin x$ (3 marks)

d Solve for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the equation

$$\sum_{k=1}^{7} \sin[(2k - 1)x] = 0.$$

(3 marks)

11 Consider $w = \dfrac{z - 1}{z + i}$ where $z = x + iy$ and $i = \sqrt{-1}$.

a If $z = i$, write w in the form $r\,\text{cis}\,\theta$. (2 marks)

b If $z = i$, find the value of w^{14}. (3 marks)

c Show that in general,

$$w = \frac{(x^2 - x + y^2 + y) + i(y - x + 1)}{x^2 + (y + 1)^2}.$$

(4 marks)

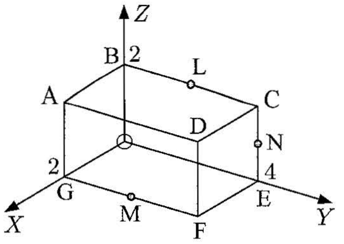
d Under what conditions is $\mathcal{R}e\,(w) = 1$? (2 marks)

e Under what conditions is w:

i real **ii** purely imaginary? (2 marks)

f Find $|z|$ given that $\arg(w) = \frac{\pi}{4}$. (2 marks)

12



D is the point $(2, 4, 2)$. L, M, and N are the midpoints of line segments [BC], [GF], and [CE] respectively.

a Find the coordinates of L, M, and N. (3 marks)

b Find $\overrightarrow{\text{ML}}$ and $\overrightarrow{\text{MN}}$. (2 marks)

c Find $\overrightarrow{\text{ML}} \times \overrightarrow{\text{MN}}$. (2 marks)

d Find the area of triangle LMN. (1 mark)

e Find the equation of plane LMN. (1 mark)

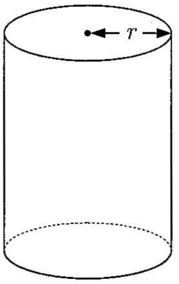
f Find the coordinates of the point P where the line segment [GC] meets the plane LMN. (2 marks)

g Find k given that $\text{L}\widehat{\text{G}}\text{E}$ measures $\arccos(k)$. (3 marks)

13 **a** A curve has equation $x^2y^3 + y^2 = x + k$ and passes through the point $(1, 2)$.

- **i** Find k. (1 mark)
- **ii** Find $\dfrac{dy}{dx}$. (2 marks)
- **iii** Find the equation of the tangent at the point $(1, 2)$. (2 marks)

b A manufacturer needs to make cylindrical containers out of tin-plate. They need an open top which will be sealed with a plastic cap. For a container of *fixed* volume we need to find the shape of the container which will minimise the amount of tin-plate used to make it. Let the radius be r and height by h.

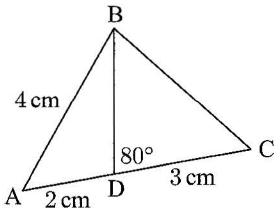


- **i** Write expressions for the volume V and outer surface area A of the can, in terms of r and h. (2 marks)
- **ii** Show that $A = \pi r^2 + 2Vr^{-1}$. (2 marks)
- **iii** Find $\dfrac{dA}{dr}$, and the relationship between r and h when $\dfrac{dA}{dr} = 0$. (3 marks)
- **iv** Prove that the surface area A will be minimised for the case in **iii**. (2 marks)
- **v** Illustrate the shape of the container which minimises the amount of tin-plate required. (1 mark)

CALCULATOR 120 marks / 3 hours

SECTION A (62 marks)

1 For the given figure:

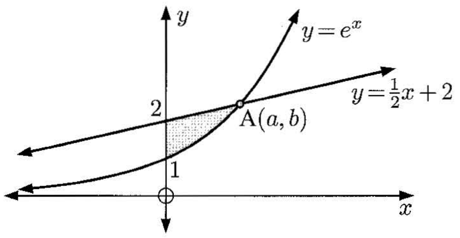


a Find the length of [BD]. (3 marks)

b Find the area of triangle BCD. (2 marks)

c Find the shortest distance from B to [DC]. (2 marks)

2



When the shaded region is rotated about the y-axis through $180°$, a solid is generated.

A(a, b) is the point of intersection of $y = e^x$ and $y = \frac{1}{2}x+2$.

- **a** Explain why the volume of the solid generated is given by $V = \pi\int_1^b (\ln y)^2\,dy - \frac{1}{3}\pi a^2(b-2)$. (2 marks)
- **b** Find a and b correct to 4 decimal places. (3 marks)
- **c** Hence, find the volume to 3 decimal places. (2 marks)

3 **a** Use mathematical induction to prove that

$$\sum_{i=1}^{n} i(i+4) = \frac{n(n+1)(2n+13)}{6}$$

for all $n \in \mathbb{Z}^+$. (6 marks)

- **b** Hence find $\sum_{i=40}^{60} i(i+4)$. (2 marks)

4 A and B are two independent events for which $\text{P}(A \cap B') = 0.2$, $\text{P}(B \cap A') = 0.3$, and $\text{P}(A \cap B) = a$.

- **a** Find the possible values of a. (3 marks)
- **b** For the larger of the possible values of a, find:
  - **i** $\text{P}(B \mid A')$ **ii** $\text{P}((A \cup B)')$. (4 marks)

5 The sum of the first four terms of an infinite geometric series is three quarters of its sum to infinity.

- **a** Find the exact value of the common ratio of the series, given that it is positive. (3 marks)
- **b** How many terms of the series need to be added to reach 99.9% of its sum to infinity? (5 marks)

6 $p(x) = x^5 - 6x^4 + ax^3 + bx^2 + cx + d$ is a real polynomial. The graph of $y = p(x)$ cuts the x-axis three times only, at -2 and $1 \pm \sqrt{2}$. The graph has y-intercept -20.

- **a** What is the nature of the other two zeros of $p(x)$? (1 mark)
- **b** Use the sum and product of the roots theorem to find the other two zeros of $p(x)$. (4 marks)
- **c** Write $p(x)$ as a product of real linear and quadratic factors. (2 marks)
- **d** For what values of x is $p(x) \geqslant 0$? (2 marks)

7 Consider the system of equations $\begin{cases} x+y-z=7 \\ 2x-y+z=11 \\ 3x+y+az=b. \end{cases}$

- **a** Write the system in augmented matrix form, and show that it reduces to $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a+1 & b-19 \end{array}\right]$. (4 marks)
- **b** If $a \neq -1$, what can be said about the solution of the original set of equations? (1 mark)
- **c** If $a = -1$ and $b = 19$, find the form of *all* solutions of the equations. (2 marks)
- **d** When does the system have no solutions? (1 mark)

8 Consider the equation $\cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$ where $\theta \in [0, 2\pi]$.

- **a** By squaring both sides of this equation, show that $\sin 2\theta = -\frac{1}{2}$. (2 marks)
- **b** Find the solutions of $\sin 2\theta = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$. (3 marks)
- **c** Show that *only two* of the solutions in **b** are solutions of $\cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$. (2 marks)
- **d** Explain why only two of the solutions in **b** were solutions to the original equation. (1 mark)

SECTION B (58 marks)

9 Consider the rational function $f(x) = \dfrac{2x-1}{x-3}$.

- **a** Find, by algebraic methods:
  - **i** the y-intercept of f (1 mark)
  - **ii** the x-intercept of f (1 mark)
  - **iii** the equations of all asymptotes of f. (2 marks)
- **b** Describe the transformations needed to transform $y = \dfrac{1}{x}$ into $y = f(x)$. (3 marks)
- **c** Draw a fully labelled graph of $y = f(x)$. (2 marks)
- **d** For what values of x is $f'(x)$:
  - **i** negative **ii** undefined? (3 marks)
- **e** Find, correct to 5 decimal places, the area bounded by f, the x-axis, and the vertical lines $x = 0$ and $x = \frac{3}{2}$. (3 marks)

10 Consider $f(x) = 4\cos^3 x - 3\cos x$.

- **a** Find *all* values of x at which $y = f(x)$ has stationary points. (4 marks)
- **b** Use technology to graph $y = f(x)$ for $x \in [-\pi, \pi]$. (1 mark)
- **c** From **b**, what do you suspect $4\cos^3 x - 3\cos x$ will simplify to? (1 mark)
- **d** Prove your result in **c** algebraically. (3 marks)
- **e** Hence:
  - **i** find an exact root of the equation $4t^3 - 3t = -\frac{1}{2}$ (3 marks)
  - **ii** find $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\cos 3x = -\cos x$. (3 marks)

11 A line in space has parametric equations

$x = 2 - t$, $y = 1 + t$, $z = 1 + 3t$, $t \in \mathbb{R}$.

A plane has equation $2x + y - 3z = 22$.

- **a** Find the point of intersection of the line and plane. (3 marks)
- **b** Find the size of the acute angle where the line meets the plane. (3 marks)
- **c** A straight line contains the point A which has position vector **a** relative to the origin O(0, 0, 0).
  The direction of the line is given by vector **b**.
  Show that the shortest distance from the origin to the line is $\sqrt{|\mathbf{a}|^2 - \dfrac{(\mathbf{a} \bullet \mathbf{b})^2}{|\mathbf{b}|^2}}$. (6 marks)
- **d** Hence find the shortest distance from the origin to the given line. (3 marks)

12 **a** When $P(x)$ is divided by $(x-a)^2$, the quotient is $Q(x)$ and the remainder is $bx+c$.

- **i** Write $P(x)$ in terms of $Q(x)$, $(x-a)^2$, and $bx+c$. (1 mark)
- **ii** Hence find $P(a)$ and $P'(a)$. (2 marks)
- **iii** Deduce that the remainder is $P'(a)(x-a)+P(a)$. (3 marks)
- **iv** Find the remainder when $P(x)=x^5$ is divided by $(x+2)^2$. (1 mark)

b Using the substitution $x=3\cos\theta$, find $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$. (6 marks)

TRIAL EXAMINATION 3

NO CALCULATOR — 120 marks / 3 hours

SECTION A (58 marks)

1 When a cubic polynomial $p(x)$ is divided by $x(2x-3)$, the remainder is $ax+b$, where a and b are real.

- **a** If the quotient is the same as the remainder, write down an expression for $p(x)$. (2 marks)
- **b** Prove that $(2x-1)$ and $(x-1)$ are both factors of $p(x)$. (1 mark)
- **c** Find in expanded form, an expression for $p(x)$, given that it has y-intercept 7 and passes through point $(2, 39)$. (3 marks)

2 Suppose $\ln\left(\dfrac{a^2}{b}\right)=k$ and $\ln\left(\dfrac{b^2}{a^3}\right)=2$.

- **a** Show that $b=e^{3k+4}$. (4 marks)
- **b** Find r and s given that $a=e^{rk+s}$. (2 marks)

3 **a** Find constants a and b given that

$$\frac{x-2}{x^2-1}=\frac{a}{x+1}+\frac{b}{x-1} \quad \text{for all } x\in\mathbb{R}. \quad \text{(3 marks)}$$

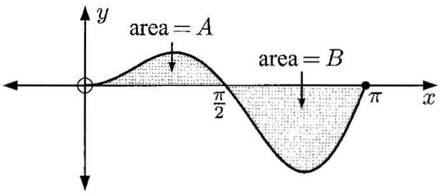
b Hence evaluate $\displaystyle\int_{-4}^{-2}\frac{x-2}{x^2-1}\,dx$ giving your answer in logarithmic form. (3 marks)

c Explain why $\displaystyle\int_{1}^{3}\frac{x-2}{x^2-1}\,dx$ does not exist. (1 mark)

4 **a** Use integration by parts to show that

$\int x\sin 2x\,dx=\frac{1}{4}\sin 2x-\dfrac{x}{2}\cos 2x+c.$ (3 marks)

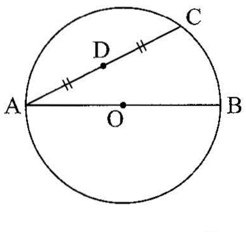
b The graph of $f(x)=x\sin 2x$, $x\in[0,\pi]$ is given below.



Show that $B=3A$. (4 marks)

5 A circle has centre O and diameter [AB]. D is the midpoint of chord [AC].

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OC}=\mathbf{c}$.



a Find vectors $\overrightarrow{AC}$ and $\overrightarrow{OD}$ in terms of **a** and **c**. (3 marks)

b Hence prove, using vector methods, that [AC] and [OD] are perpendicular. (3 marks)

c What theorem has been proved in **b**? (1 mark)

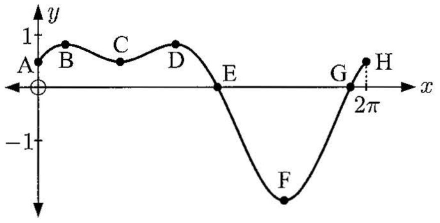
6 Let $f(x)=x^2+4x$, $x\in\,]-\infty,-2]$ and $g(x)=\sqrt{3-2x}$. Find:

- **a** the domain of $g(x)$ (1 mark)
- **b** $(g\circ f)(-3)$ (2 marks)
- **c** $f^{-1}(x)$. (3 marks)

7 Solve for x:

- **a** $2^{2x}+2^{x+1}=15$ (3 marks)
- **b** $\sin^2 x+\cos x=1.25$, $x\in[-\pi,\pi]$ (3 marks)

8 The graph of $f(x)=\frac{1}{2}\cos 2x+\sin x$ for $x\in[0,2\pi]$ is illustrated, but not drawn to scale.



- **a** State the coordinates of A and H. (2 marks)
- **b** Find the exact coordinates of the stationary points at B, C, D, and F. (5 marks)
- **c** Find the exact values of the x-intercepts at E and G. (6 marks)

SECTION B (62 marks)

9 **a** Show that:

- **i** $\cos(A+B)+\cos(A-B)=2\cos A\cos B$
- **ii** $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ (3 marks)

b Hence show that:

- **i** $\cos 3\theta+\cos\theta=2\cos 2\theta\cos\theta$
- **ii** $\sin 3\theta+\sin\theta=2\sin 2\theta\cos\theta$ (3 marks)

c If $z=\operatorname{cis}\theta$, what are z^2 and z^3? (1 mark)

d Use **b** to show that $z^3+z=2\cos\theta\operatorname{cis}2\theta$. (3 marks)

e Hence find $\left|z^3+z\right|$ and $\arg(z^3+z)$. (1 mark)

f On an Argand diagram with $\theta\approx 25°$, illustrate z, z^2, z^3, and z^3+z.

Show that $\arg(z^3+z)=\arg(z^2)$. (3 marks)

g Use your Argand diagram to find all values of θ, where $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, such that $z+z^3$ is purely imaginary. (2 marks)

10 **a** **i** If $f(x)=\tan^3 x$, find $f'(x)$ in terms of $\sec x$ only. (2 marks)

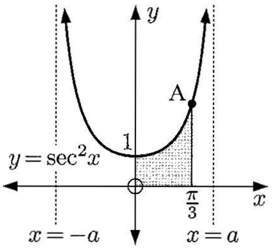
ii Hence show that

$\int\sec^4 x\,dx=\tan x+\frac{1}{3}\tan^3 x+c.$ (3 marks)

b One part of the graph of $y=\sec^2 x$ is shown.

Find:

- **i** the value of a
- **ii** the coordinates of A.



(2 marks)

c Find the exact value of the area of the shaded region. (2 marks)

d If the shaded region is rotated about the x-axis through $360°$, find the volume of the solid generated. (3 marks)

e If the shaded region is rotated about the y-axis through $360°$, explain why the volume of the solid generated is $\frac{4\pi^3}{9} - \pi \int_1^4 \left[\arccos\left(\frac{1}{\sqrt{y}}\right)\right]^2 dy.$ (4 marks)

11 Suppose $\mathbf{r} = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}$, $t \in \mathbb{R}$ is an equation of line L.

The plane P has a normal vector $3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and passes through the point A$(-1, 0, 4)$.

a Show that the point B$(9, -5, 2)$ lies on the line L. (1 mark)

b Give an equation of the plane P. (2 marks)

c Show that the line L meets the plane P at the point C$(1, 3, -2)$. (2 marks)

d The line N through the point B$(9, -5, 2)$ is perpendicular to the plane P. Find an equation of the line N. (3 marks)

e Show that the point of intersection of the line N and the plane P is the point D$(3, 3, 4)$. (2 marks)

f Find the coordinates of the point B$'$ on the line N such that the plane P bisects the line segment [BB$'$]. (2 marks)

g Decide if the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is parallel to the line (CB$'$). Give reasons for your answer. (2 marks)

12 **a** Use mathematical induction to prove that $2^{4n+3} + 3^{3n+1}$ is divisible by 11 for all $n \in \mathbb{Z}^+$. (8 marks)

b a, b, and c are consecutive terms of an arithmetic sequence. a, $b+1$, and $c+29$ are consecutive terms of a geometric sequence.
Given that $a + b + c = 33$, find all possible values for a, b, and c. (8 marks)

CALCULATOR 120 marks / 3 hours

SECTION A (58 marks)

1 **a** If $z = r \operatorname{cis} \theta$, find in terms of r and θ:

i z^3 **ii** $\sqrt[3]{z}$ (2 marks)

b Given $-11 + ai = (1 - ai)^3$ where $a \in \mathbb{R}$, find the possible values of a. (4 marks)

2 Suppose $f(x) = \begin{cases} \frac{1}{2}e^{-bx}, & 0 \leqslant x \leqslant 1, \quad b \neq 0 \\ 0, & \text{otherwise.} \end{cases}$

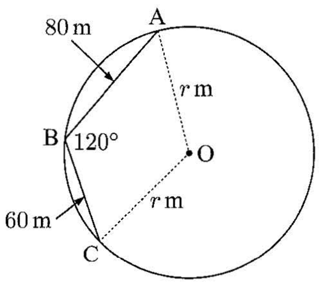
a Show that if $f(x)$ is a well defined PDF, then $e^{-b} + 2b - 1 = 0$. (3 marks)

b Find b correct to 3 decimal places. (1 mark)

c Find the mean μ of the distribution of X. (1 mark)

d Calculate Var(X). (2 marks)

3 A, B, and C are lamp posts on the edge of a circular lake of radius r metres. O is the lake's centre.



a Use triangle ABC to find the value of $(\text{AC})^2$. (2 marks)

b Use triangle AOC to find the value of $(\text{AC})^2$. (2 marks)

c Calculate the radius of the lake. (1 mark)

d Find the measure of $\text{B}\widehat{\text{A}}\text{O}$. (2 marks)

4 A sample of 20 randomly selected cockles is taken from a population of thousands of cockles, with no replacement. If a cockle's shell is cracked, it is unsaleable. It is known that 3.2% of shells in the population are cracked.

a Explain why this sampling method is not strictly binomial, but the binomial model gives a very good approximation. (2 marks)

b Suppose X = the number of cracked shells in the sample of 20 cockles. Find:

i $\text{P}(X \leqslant 2)$ **ii** $\text{P}(X \geqslant 4)$. (4 marks)

5 **a** By finding $\frac{d}{dx}(x^{n+1} \ln x)$, show that

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2}((n+1)\ln x - 1) + c$$

provided $n \neq -1$. (5 marks)

b Find $\int x^n \ln x \, dx$ when $n = -1$. (2 marks)

6 The displacement of a particle moving in a straight line is given by $s = t \sin\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{2}\right)$ metres, where t is the time in seconds.

a Find an expression for v, the velocity of the particle at time t, in metres per second. (2 marks)

b The particle starts at rest. When does the particle first change direction, and what is its position at that time? (3 marks)

c Find the exact value of the acceleration of the particle after $\frac{\pi}{3}$ seconds. (3 marks)

7 **a** Find the *exact* value of x if
$\arctan\left(\frac{x}{3}\right) + \arctan 6 = \arctan 3.$ (5 marks)

b If $y = \arctan\left(\frac{x}{3}\right)$, find $\frac{dy}{dx}$. (3 marks)

8 A circle with equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(0, 3)$, $(2, -1)$, and $(8, 7)$.

a Find three equations involving a, b, and c only. (3 marks)

b Write the equations in augmented matrix form. Simplify the result to echelon form. (3 marks)

c Find the values of a, b, and c. (3 marks)

SECTION B **(62 marks)**

9 The results obtained in a Science examination are distributed approximately normally with mean 56.7 and standard deviation 18.2. All final results are integers.

a A student who did the Science examination is selected at random. Find the probability that the student had a result:

i between 65 and 85 inclusive

ii at least 70. (2 marks)

b An 'A' grade is awarded to a student whose result is in the top 10%.

An 'F' grade is given to a student whose result is in the bottom 15%.

Determine the:

i smallest result for which an 'A' is awarded

ii largest result for which an 'F' is given. (4 marks)

c Four of the students are chosen at random. What is the probability that two will get an 'A' and the other two will get an 'F'? (3 marks)

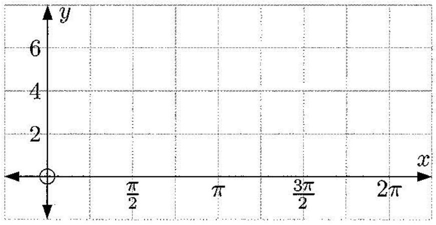
d Suppose 20 of the students are chosen at random. Find the probability that:

i none receive an 'A'

ii at least three receive an 'A'. (3 marks)

10 **a** Consider the function $f(x) = \dfrac{12 + 2\sin 2x}{3 - \sin 2x}$.

i Sketch the graph of $y = f(x)$ for $x \in [0,\ 2\pi]$ using the given grid:



(3 marks)

ii Find the exact coordinates of any maxima turning points. (4 marks)

b The formula $h(x) = \dfrac{a + b\sin 2x}{c - \sin 2x}$ gives the profile of plastic sheeting used on the tops of verandahs. The constants a, b, and c are positive.

i Show that $h'(x) = \dfrac{2\cos 2x(a + bc)}{(c - \sin 2x)^2}$. (3 marks)

ii Hence find the maximum (M) and minimum (m) values of $h(x)$. (2 marks)

iii If the height of the profile is H, show that $H = \dfrac{2(a + bc)}{c^2 - 1}$. (2 marks)

iv Check the formula in **iii** using the specific example in part **a**. (2 marks)

11 L_1 and L_2 are two lines in space given by:

$$L_1\text{:}\quad \frac{x+1}{2} = -y = \frac{z-1}{2} = \lambda$$

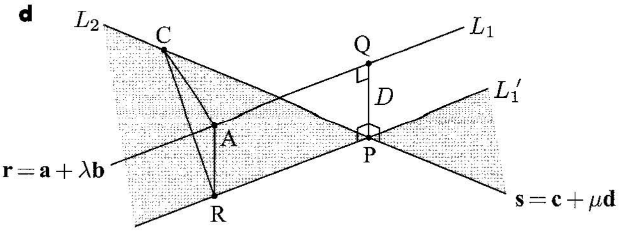
$$L_2\text{:}\quad \frac{x-1}{3} = 1 - y = \frac{z-2}{2} = \mu.$$

a Write the equations of L_1 and L_2 in vector form. (2 marks)

b Show that L_1 and L_2 do not intersect, and also that they are not parallel. (5 marks)

c Find a vector which is perpendicular to both L_1 and L_2. (2 marks)

d



L_1 and L_2 are two skew lines in space. There exist points P on L_2 and Q on L_1, where PQ is the shortest distance between lines L_1 and L_2. When this occurs, [PQ] is perpendicular to both L_1 and L_2.

We now translate L_1 to L_1', where L_1' passes through P, and L_2 and L_1' form the shaded plane.

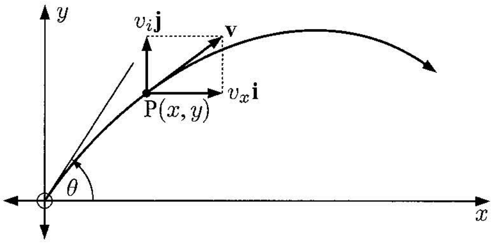
A and C are two fixed points on L_1 and L_2 respectively.

i Explain why the vector in the direction of [PQ] is $\mathbf{b} \times \mathbf{d}$. (1 mark)

ii In triangle CAR let $\widehat{CAR} = \theta$. Hence show that the shortest distance between L_1 and L_2 is $\dfrac{|(\mathbf{c} - \mathbf{a}) \bullet (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$. (4 marks)

e Find the shortest distance between the original lines L_1 and L_2. (3 marks)

12



A missile is launched at an angle θ to the horizontal ground. At the point $\mathrm{P}(x, y)$ on its path, the horizontal component of its velocity is $v_x = v_0\cos\theta$, and the vertical component is $v_y = v_0\sin\theta - gt$. v_0 is the initial speed of the missile, and g is the gravitational constant.

a Show that at P,
$x = (v_0\cos\theta)t$ and $y = (v_0\sin\theta)t - \frac{1}{2}gt^2$. (4 marks)

b By eliminating t, show that $y = (\tan\theta)x - (\sec^2\theta)\dfrac{gx^2}{2v_0^2}$. (2 marks)

c What is the nature of the missile's path? Give evidence to support your answer. (1 mark)

d Show that the missile reaches its maximum height when $x = \dfrac{v_0^2\sin 2\theta}{2g}$, and that the maximum height is $\dfrac{v_0^2\sin^2\theta}{2g}$. (4 marks)

e Find the range of the missile, which is the horizontal distance it will travel before landing. (1 mark)

f What value of θ will achieve maximum range for the missile? (1 mark)

g Suppose $v_0 = 300\ \mathrm{m\,s^{-1}}$, $\theta = 45°$, and $g = 9.81\ \mathrm{m\,s^{-2}}$. Find:

i the maximum height reached

ii the range of the missile. (2 marks)

h SPX3 rockets have an intitial velocity of 400 $\mathrm{m\,s^{-1}}$. If the target is 9.5 km away, at what angle must the missile be fired at in order to hit the target? (2 marks)

SOLUTIONS TO TOPIC 1 (ALGEBRA)

1 **a** $18+16+14+12+\ldots$ is an arithmetic series with $u_1=18,\ d=-2$ and $n=30$.

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1+(n-1)d) \\ \therefore\ S_{30} &= \tfrac{30}{2}(2\times 18+29\times(-2)) \\ &= -330 \end{aligned}$$

b $48+24+12+6+\ldots$ is a geometric series with $u_1=48$, $r=\frac{1}{2}$ and $n=30$.

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ \therefore\ S_{30} &= \frac{48\left(1-(\frac{1}{2})^{30}\right)}{1-\frac{1}{2}} \approx 96.0 \end{aligned}$$

2 $$\begin{aligned} \frac{x^a\sqrt{x^{3a}}}{x^{-2a}} &= x^a(x^{3a})^{\frac{1}{2}}\times x^{-(-2a)} \\ &= x^a\times x^{\frac{3a}{2}}\times x^{2a} \\ &= x^{\frac{9a}{2}} \end{aligned}$$

3 **a** $$\begin{aligned} \log_4 8 &= \frac{\log 8}{\log 4} \\ &= \frac{3\log 2}{2\log 2} \\ &= \tfrac{3}{2} \end{aligned}$$

b $$\begin{aligned} \log_9\left(\tfrac{1}{27}\right) &= \frac{\log\left(\frac{1}{27}\right)}{\log 9} \\ &= \frac{-3\log 3}{2\log 3} \\ &= -\tfrac{3}{2} \end{aligned}$$

c $$\begin{aligned} \log_{\frac{1}{3}}\sqrt{3} &= \frac{\log\sqrt{3}}{\log\frac{1}{3}} \\ &= \frac{\frac{1}{2}\log 3}{-\log 3} \\ &= -\tfrac{1}{2} \end{aligned}$$

4 $$\begin{aligned} (2-ai)^3 &= (2-ai)^2(2-ai) \\ &= (4-2\times 2ai+(ai)^2)(2-ai) \\ &= (4-4ai-a^2)(2-ai) \\ &= 8-4ai-8ai+4a^2i^2-2a^2+a^3i \\ &= 8-12ai-4a^2-2a^2+a^3i \\ &= (8-6a^2)+i(a^3-12a) \end{aligned}$$

5 $$\begin{aligned} T_{r+1} &= \binom{12}{r}(2x)^{12-r}\left(\frac{-1}{x^2}\right)^r \\ &= \binom{12}{r}2^{12-r}x^{12-r}(-1)^r x^{-2r} \\ &= \binom{12}{r}2^{12-r}(-1)^r x^{12-3r} \end{aligned}$$

a $$\begin{aligned} 12-3r &= 3 \\ \therefore\ r &= 3 \\ \therefore\ T_4 &= \binom{12}{3}2^{12-3}(-1)^3x^3 \\ &= -112\,640x^3 \end{aligned}$$

The coefficient of x^3 is $-112\,640$.

b $$\begin{aligned} 12-3r &= 0 \\ \therefore\ r &= 4 \\ \therefore\ T_5 &= \binom{12}{4}2^{12-4}(-1)^4x^0 \\ &= 126\,720 \end{aligned}$$

The constant term is 126 720.

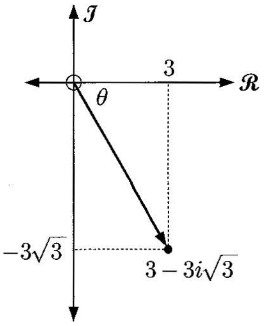
6 $$\begin{aligned} u_1 = 27 &\text{ and } u_4 = 8 \\ \therefore\ u_1r^3 &= 8 \\ \therefore\ r^3 &= \tfrac{8}{27} \\ \therefore\ r &= \tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} \therefore\ \text{the series sum } S &= \frac{u_1}{1-r} \\ &= \frac{27}{1-\frac{2}{3}} \\ &= 81 \end{aligned}$$

7 $$\begin{aligned} &\left(\frac{3x^{-1}}{2a^2}\right)^{-2}\times\left(\frac{4x^2}{27a^{-3}}\right)^{-1} \\ &= \left(\frac{2a^2}{3x^{-1}}\right)^2\times\left(\frac{27a^{-3}}{4x^2}\right) \\ &= \left(\frac{4a^4}{9x^{-2}}\right)\times\left(\frac{27a^{-3}}{4x^2}\right) \\ &= \frac{108a}{36} \\ &= 3a \end{aligned}$$

8 $$\begin{aligned} \log_5(2x-1) &= -1 \\ \therefore\ 5^{-1} &= 2x-1 \\ \therefore\ \tfrac{1}{5} &= 2x-1 \\ \therefore\ \tfrac{6}{5} &= 2x \\ \therefore\ x &= \tfrac{3}{5} \end{aligned}$$

9 $$\begin{aligned} &3-3i\sqrt{3} \\ &= 6\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) \\ &= 6\left(\cos\left(-\tfrac{\pi}{3}\right)+i\sin\left(-\tfrac{\pi}{3}\right)\right) \end{aligned}$$



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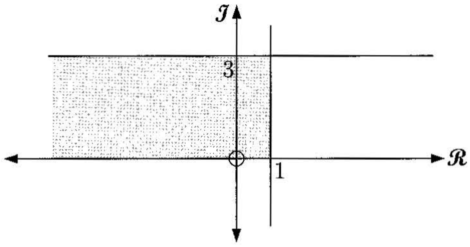


11 $$\begin{aligned} \binom{n}{r}+\binom{n}{r-1} &= \frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-(r-1))!} \\ &= \frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n+1-r)!} \\ &= \frac{n!(n+1-r)}{r!(n+1-r)!}+\frac{n!r}{r!(n+1-r)!} \\ &= \frac{n!(n+1-r)+n!r}{r!(n+1-r)!} \\ &= \frac{n!(n+1)-n!r+n!r}{r!(n+1-r)!} \\ &= \frac{(n+1)!}{r!\,((n+1)-r)!} \\ &= \binom{n+1}{r} \text{ for all } n,\ r\in\mathbb{Z}^+,\ r\leqslant n \end{aligned}$$

12 **a** $$\begin{aligned} u_n &= \frac{2n+1}{3} \\ \therefore\ u_{n+1}-u_n &= \frac{2(n+1)+1}{3}-\frac{2n+1}{3} \\ &= \frac{2n+3-2n-1}{3} \\ &= \tfrac{2}{3} \text{ for all } n\in\mathbb{Z}^+ \end{aligned}$$

$\therefore$ consecutive terms differ by $\frac{2}{3}$, so the sequence is arithmetic with $d=\frac{2}{3}$.

b $u_1=\frac{2(1)+1}{3}=1$ and $d=\frac{2}{3}$

Now $u_n=u_1+(n-1)d$

$$\begin{aligned} \therefore\ u_{50} &= 1+49\times\tfrac{2}{3} = 33\tfrac{2}{3} \\ S_n &= \frac{n}{2}(u_1+u_n) \\ \therefore\ S_{50} &= \tfrac{50}{2}(1+33\tfrac{2}{3}) = 866\tfrac{2}{3} \end{aligned}$$

c
$$\begin{aligned} \text{Let } u_n &= 117 \\ \therefore\ u_1 + (n-1)d &= 117 \\ \therefore\ 1 + (n-1)\tfrac{2}{3} &= 117 \\ \therefore\ n - 1 &= 174 \\ \therefore\ n &= 175 \end{aligned}$$

So, 117 *is* a term of the sequence. It is the 175th term.

d **i**
$$\begin{aligned} \sum_{n=1}^{40} u_n &= S_{40} \\ S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore\ S_{40} &= \tfrac{40}{2}(2 \times 1 + 39 \times \tfrac{2}{3}) \\ &= 560 \end{aligned}$$

ii
$$\begin{aligned} \sum_{n=30}^{60} u_n &= S_{60} - S_{29} \\ &= \tfrac{60}{2}\left(2(1) + 59(\tfrac{2}{3})\right) - \tfrac{29}{2}\left(2(1) + 28(\tfrac{2}{3})\right) \\ &= 1240 - 299\tfrac{2}{3} \\ &= 940\tfrac{1}{3} \end{aligned}$$

13
$$\begin{aligned} 8^{2x-3} &= 16^{2-x} \\ \therefore\ (2^3)^{2x-3} &= (2^4)^{2-x} \\ \therefore\ 2^{6x-9} &= 2^{8-4x} \\ \therefore\ 6x - 9 &= 8 - 4x \\ \therefore\ 10x &= 17 \\ \therefore\ x &= \tfrac{17}{10} \end{aligned}$$

14
$$\begin{aligned} \log_b A &= \frac{\log_c A}{\log_c b} \\ \therefore\ \log_5 9 &= \frac{\log_3 9}{\log_3 5} \\ \therefore\ \frac{8}{\log_5 9} &= \frac{8}{\frac{\log_3 9}{\log_3 5}} \\ &= \frac{8 \log_3 5}{\log_3 9} \\ &= \frac{8 \log_3 5}{2} \\ &= 4 \log_3 5 \end{aligned}$$

15
$$\begin{aligned} \left(\cos(\tfrac{2\pi}{3}) - i\sin(\tfrac{2\pi}{3})\right)^{10} &= \left(\cos(-\tfrac{2\pi}{3}) + i\sin(-\tfrac{2\pi}{3})\right)^{10} \\ &= \left(\text{cis}\,(-\tfrac{2\pi}{3})\right)^{10} \\ &= \text{cis}\left(-\tfrac{20\pi}{3}\right) \\ &= \cos\left(-\tfrac{20\pi}{3}\right) + i\sin\left(-\tfrac{20\pi}{3}\right) \\ &= \cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right) \\ &= -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2} \end{aligned}$$

16 There are 12 numbers up for selection and we choose 3.
$\therefore$ the total number of combinations is $C_3^{12} = 220$.

17
$$\begin{aligned} (1+x)^n = \tbinom{n}{0} \times 1^n \times x^0 + \tbinom{n}{1} \times 1^{n-1} \times x^1 \\ + \tbinom{n}{2} \times 1^{n-2} \times x^2 + \ldots. + \tbinom{n}{n} \times 1^0 \times x^n \end{aligned}$$

Now set $x = 1$:

$$\begin{aligned} (1+1)^n = \tbinom{n}{0} \times 1^n \times 1^0 + \tbinom{n}{1} \times 1^{n-1} \times 1^1 \\ + \tbinom{n}{2} \times 1^{n-2} \times 1^2 + \ldots. + \tbinom{n}{n} \times 1^0 \times 1^n \end{aligned}$$

$$\therefore\ 2^n = \tbinom{n}{0} + \tbinom{n}{1} + \tbinom{n}{2} + \ldots. + \tbinom{n}{n}$$

$$\therefore\ 2^n = \sum_{r=0}^{n} \tbinom{n}{r}$$

18 **a**
$$\begin{aligned} \frac{u_1}{1-r} &= 1.5 \\ \text{and } u_1 &= 1 \\ \therefore\ 1 - r &= \tfrac{1}{1.5} \\ \therefore\ 1 - r &= \tfrac{2}{3} \\ \therefore\ r &= \tfrac{1}{3} \end{aligned}$$

b
$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ \therefore\ S_7 &= \frac{u_1(1-r^7)}{1-r} \\ &= \frac{1\left(1 - (\frac{1}{3})^7\right)}{1 - \frac{1}{3}} \\ &= \tfrac{3}{2}\left(1 - \tfrac{1}{2187}\right) \\ &= \tfrac{1093}{729} \end{aligned}$$

19
$$\begin{aligned} &\frac{3^{x+1} - 3^x}{2(3^x) - 3^{x-1}} \\ &= \frac{3^{x-1}(3^2 - 3)}{3^{x-1}(2 \times 3 - 1)} \\ &= \tfrac{6}{5} \end{aligned}$$

20
$$\begin{aligned} &2\ln x + \ln(x-1) - \ln(x-2) \\ &= \ln x^2 + \ln(x-1) - \ln(x-2) \\ &= \ln\left(\frac{x^2(x-1)}{(x-2)}\right) \end{aligned}$$

21 **a**
$$\begin{aligned} &\frac{3+4i}{1-3i} \times \left(\frac{1+3i}{1+3i}\right) \\ &= \frac{(3+4i)(1+3i)}{(1-3i)(1+3i)} \\ &= \frac{3 + 9i + 4i + 12i^2}{1 - 9i^2} \\ &= \frac{3 + 13i - 12}{1 - 9(-1)} \\ &= \frac{-9 + 13i}{10} \\ &= -\tfrac{9}{10} + \tfrac{13}{10}i \end{aligned}$$

b
$$\begin{aligned} &\frac{3}{i}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)^2 \\ &= \frac{3}{i}\left(\frac{1-2i}{\sqrt{5}}\right)^2 \\ &= \frac{3}{i}\left(\frac{(1-2i)^2}{5}\right) \\ &= \frac{3}{i}\left(\frac{1 - 4i + 4i^2}{5}\right) \\ &= \frac{3}{i}\left(\frac{-3-4i}{5}\right) \\ &= \frac{-9 - 12i}{5i} \\ &= \frac{-9i - 12i^2}{5i^2} \\ &= \frac{-9i + 12}{-5} \\ &= -\tfrac{12}{5} + \tfrac{9}{5}i \end{aligned}$$

22
$$\begin{aligned} T_{r+1} &= \tbinom{n}{r}(3x)^{n-r}(2)^r \\ &= \tbinom{n}{r} \times 3^{n-r} \times 2^r \times x^{n-r} \end{aligned}$$

Coefficient of x^3: $n - r = 3$

$$\begin{aligned} \therefore\ r &= n - 3 \\ T_{n-2} &= \tbinom{n}{n-3} \times 3^3 \times 2^{n-3} \times x^3 \end{aligned}$$

Coefficient of x: $n - r = 1$

$$\begin{aligned} \therefore\ r &= n - 1 \\ T_n &= \tbinom{n}{n-1} \times 3 \times 2^{n-1} \times x \end{aligned}$$

Now $\left[\tbinom{n}{n-1} \times 3 \times 2^{n-1}\right] \times 21 = \tbinom{n}{n-3} \times 3^3 \times 2^{n-3}$

$$\begin{aligned} \therefore\ \frac{n!}{(n-1)!\,(n-(n-1))!} \times 3^2 \times 2^{n-1} \times 7 \\ = \frac{n!}{(n-3)!\,(n-(n-3))!} \times 3^3 \times 2^{n-3} \end{aligned}$$

$$\begin{aligned} \therefore\ n \times 3^2 \times 2^{n-1} \times 7 &= \frac{n(n-1)(n-2)}{6} \times 3^3 \times 2^{n-3} \\ \therefore\ n \times 2^2 \times 7 &= \frac{n(n-1)(n-2)}{6} \times 3 \\ \therefore\ 56n &= n(n-1)(n-2) \\ \therefore\ 56n &= n(n^2 - 3n + 2) \\ \therefore\ 56n &= n^3 - 3n^2 + 2n \\ \therefore\ n^3 - 3n^2 - 54n &= 0 \\ \therefore\ n(n^2 - 3n - 54) &= 0 \\ \therefore\ n(n-9)(n+6) &= 0 \\ \therefore\ n &= 9 \quad \{n \in \mathbb{Z}^+\} \end{aligned}$$

23 Writing the system in augmented matrix form,

$$\left[\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 1 & 2 & -3 & 1 \\ -1 & -1 & 1 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 0 & 6+a & -11 & -1 \\ 0 & 1 & -2 & 13 \end{array}\right] \begin{array}{l} \\ R_2 \to 3R_2 - R_1 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -a & 2 & 4 \\ 0 & 6+a & -11 & -1 \\ 0 & 0 & -2(6+a)+11 & 13(6+a)+1 \end{array}\right]$$

$$R_3 \to (6+a)R_3 - R_2$$

From the last line, there is a unique solution when

$$-2(6+a)+11 \neq 0$$
$$\therefore\ -2a \neq 1$$
$$\therefore\ a \neq -\tfrac{1}{2}$$

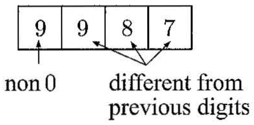
24
$$\frac{z+2}{z-2} = i$$
$$\therefore\ z+2 = i(z-2)$$
Let $z = a+bi$
$$\therefore\ (a+bi)+2 = i\left((a+bi)-2\right)$$
$$\therefore\ (2+a)+bi = ai+bi^2-2i$$
$$\therefore\ (2+a)+bi = -b+(a-2)i$$

Equating real and imaginary parts:

$$2+a = -b \quad .... (1)$$
$$\text{and } b = a-2 \quad .... (2)$$

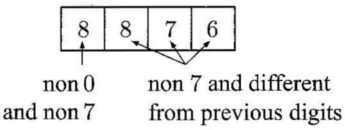
Substituting (2) into (1): $2+a = 2-a$
$$\therefore\ 2a = 0$$
$$\therefore\ a = 0$$
$$\therefore\ b = -2$$
So, $z = -2i$.

25 **a**



$\therefore$ there are $9 \times 9 \times 8 \times 7 = 4536$ numbers

b Numbers that do *not* have a "7" as one of the four digits:



$\therefore$ there are $8 \times 8 \times 7 \times 6 = 2688$ numbers that do not contain a "7".

$\therefore$ there are $4536 - 2688 = 1848$ numbers that *do* contain a "7".

26 P_n is: "$1 \times 2 + 2 \times 5 + 3 \times 8 + ... + n(3n-1) = n^2(n+1)$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2 = 2$ and RHS $= 1^2(1+1) = 2$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$1 \times 2 + 2 \times 5 + 3 \times 8 + ... + k(3k-1) = k^2(k+1)$

Thus $1 \times 2 + 2 \times 5 + 3 \times 8 + ... + k(3k-1)$
$$+ (k+1)\,(3(k+1)-1)$$
$$= k^2(k+1) + (k+1)\,(3(k+1)-1) \quad \{\text{using } P_k\}$$
$$= (k+1)\left(k^2 + 3(k+1) - 1\right)$$
$$= (k+1)(k^2+3k+2)$$
$$= (k+1)(k+1)(k+2)$$
$$= (k+1)^2\,([k+1]+1)$$

Thus P_{k+1} is true when P_k is true. Since P_1 is true, P_n is true for $n \in \mathbb{Z}^+$.

{Principle of mathematical induction}

27 **a**
$$u_7 = 1 \quad \text{and} \quad u_{15} = -23$$
$$\therefore\ u_{15} - u_7 = -23 - 1$$
$$\therefore\ (u_1 + 14d) - (u_1 + 6d) = -24$$
$$\therefore\ 8d = -24$$
$$\therefore\ d = -3$$
$$\text{and } u_1 = 1 - 6d$$
$$= 19$$
$$\text{So, } u_{27} = u_1 + 26d$$
$$= 19 + 26(-3)$$
$$= -59$$

b
$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore\ S_{27} = \tfrac{27}{2}(u_1 + u_{27})$$
$$= \tfrac{27}{2}(19 - 59)$$
$$= -540$$

28
$$4^x + 4 = 17(2^{x-1})$$
$$\therefore\ 2^{2x} - 17(2^{x-1}) + 4 = 0$$
$$\therefore\ 2 \times 2^{2x} - 17(2^x) + 8 = 0$$
$$\therefore\ 2X^2 - 17X + 8 = 0 \quad \{\text{letting } X = 2^x\}$$
$$\therefore\ (2X-1)(X-8) = 0$$
$$\therefore\ X = \tfrac{1}{2} \text{ or } 8$$
$$\therefore\ 2^x = 2^{-1} \text{ or } 2^3$$
$$\therefore\ x = -1 \text{ or } 3$$

29 **a**
$$M = ab^3$$
$$\therefore\ \log_b M = \log_b(ab^3)$$
$$\therefore\ \log_b M = \log_b a + \log_b b^3$$
$$\therefore\ \log_b M = \log_b a + 3$$

b
$$D = \frac{a}{b^2}$$
$$\therefore\ \log_b D = \log_b\left(\frac{a}{b^2}\right)$$
$$\therefore\ \log_b D = \log_b a - \log_b b^2$$
$$\therefore\ \log_b D = \log_b a - 2$$

30 Let $z = a+bi$, so $z^* = a-bi$, $a, b \in \mathbb{R}$.
$$\text{Now, } z^2 = (z^*)^2$$
$$\text{So } (a+bi)^2 = (a-bi)^2$$
$$a^2 + 2abi + b^2i^2 = a^2 - 2abi + b^2i^2$$
$$(a^2 - b^2) + 2abi = (a^2 - b^2) - 2abi$$

Equating imaginary parts gives

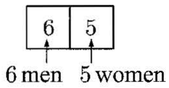
$$2ab = -2ab$$
$$\therefore\ 4ab = 0$$
$$\therefore\ a = 0 \text{ or } b = 0.$$

So, $z = a$ or $z = bi$, $a, b \in \mathbb{R}$.

$\therefore$ z is either real or purely imaginary.

31 **a** There are $\binom{11}{2} = 55$ ways of choosing 2 people from 11.

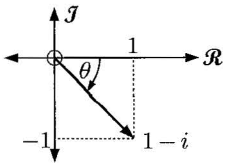
b



There are $6 \times 5 = 30$ handshakes between a man and a woman.

32 $|1-i| = \sqrt{1^2+(-1)^2}$

$= \sqrt{2}$



$\theta = \frac{\pi}{4}$

$\therefore \arg(1-i) = -\frac{\pi}{4}$

$\therefore 1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$$\begin{aligned}\therefore (1-i)^{11} &= \left[\sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)\right]^{11}\\ &= (\sqrt{2})^{11}\text{cis}\left(-\tfrac{11\pi}{4}\right) \quad \text{\{De Moivre's theorem\}}\\ &= 32\sqrt{2}\,\text{cis}\left(-\tfrac{3\pi}{4}\right)\\ &= 32\sqrt{2}\left[\cos\left(-\tfrac{3\pi}{4}\right)+i\sin\left(-\tfrac{3\pi}{4}\right)\right]\\ &= 32\sqrt{2}\left[-\tfrac{1}{\sqrt{2}}+i\left(-\tfrac{1}{\sqrt{2}}\right)\right]\\ &= -32-32i\end{aligned}$$

33 **a** $$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{12\left(\frac{2}{3}\right)^n}{12\left(\frac{2}{3}\right)^{n-1}}\\ &= \tfrac{2}{3} \quad \text{for all } n \in \mathbb{Z}^+\end{aligned}$$

So, consecutive terms have a common ratio of $\frac{2}{3}$.

Thus, the sequence is geometric with $r = \frac{2}{3}$.

b $$\begin{aligned}u_5 &= 12\left(\tfrac{2}{3}\right)^4\\ &= 12\left(\tfrac{16}{81}\right)\\ &= \tfrac{64}{27}\end{aligned}$$

c **i** $$\begin{aligned}\sum_{n=1}^{\infty} u_n &= \frac{u_1}{1-r}\\ &= \frac{12\left(\frac{2}{3}\right)^0}{1-\frac{2}{3}}\\ &= \frac{12}{\frac{1}{3}}\\ &= 36\end{aligned}$$

ii $$\sum_{n=1}^{20} u_n = S_{20}$$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore S_{20} = \frac{12\left(1-\left(\frac{2}{3}\right)^{20}\right)}{1-\frac{2}{3}}$$

$$\approx 35.9892$$

34 $2^a 8^b = \frac{1}{2}$ and $\frac{3^{-a}}{3^{b+1}} = 9$

$\therefore 2^a(2^3)^b = 2^{-1}$ and $3^{-a}3^{-(b+1)} = 3^2$

$\therefore 2^{a+3b} = 2^{-1}$ and $3^{-a-b-1} = 3^2$

So $a+3b = -1$ .... (1)

$-a-b-1 = 2$ .... (2)

From (2), $a = -3-b$

Substituting into (1), $-3-b+3b = -1$

$\therefore 2b = 2$

$\therefore b = 1$

$\therefore a = -4$

35 **a** $1 = \text{cis}\ 0 = \text{cis}\,(0+k2\pi)$ for all $k \in \mathbb{Z}$

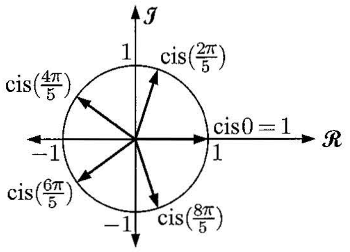
So $z^5 = \text{cis}\,(k2\pi)$

$\therefore z = [\text{cis}\,(k2\pi)]^{\frac{1}{5}}$

$\therefore z = \text{cis}\left(\frac{k2\pi}{5}\right)$ {De Moivre's theorem}

$\therefore z = \text{cis}\ 0,\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{4\pi}{5}\right),\ \text{cis}\left(\frac{6\pi}{5}\right),$

$\text{cis}\left(\frac{8\pi}{5}\right)$ {letting $k = 0, 1, 2, 3, 4$}



b $$\begin{aligned}w &= \text{cis}\left(\tfrac{2\pi}{5}\right)\\ \text{Now } w^5 &= 1\\ \therefore w^5 - 1 &= 0\\ \therefore (w-1)(w^4+w^3+w^2+w+1) &= 0\\ \therefore w^4+w^3+w^2+w+1 &= 0 \quad \{w \neq 1\}\end{aligned}$$

36 $$\begin{aligned}&\cos 3\theta + i\sin 3\theta\\ &= \text{cis}\,3\theta\\ &= (\text{cis}\,\theta)^3 \quad \text{\{De Moivre's theorem\}}\\ &= (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\\ &= [\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta]\end{aligned}$$

Equating imaginary parts,

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

37 $\log_3 x + \log_3(x-2) = 1$

$\therefore \log_3(x(x-2)) = 1$

$\therefore 3^1 = x(x-2)$

$\therefore 3 = x^2 - 2x$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x-3)(x+1) = 0$

$\therefore x = 3 \quad \{x > 2\}$

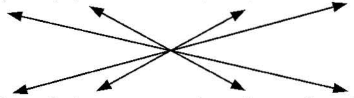
38 Consider the x^n term in: $(1+x)^{2n} = (1+x)^n(1+x)^n$

On the LHS, $T_{n+1} = \binom{2n}{n}1^n x^n$, so x^n has coefficient $\binom{2n}{n}$.

On the RHS, we have

$(1+x)^n(1+x)^n$

$= \left[\binom{n}{0} + \binom{n}{1}x + \ldots. + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$



$\times\left[\binom{n}{0} + \binom{n}{1}x + \ldots. + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$

$\therefore$ the coefficient of x^n

$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots. + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$

$= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \ldots. + \binom{n}{n-1}\binom{n}{n-1} + \binom{n}{n}\binom{n}{n}$

$\left\{\binom{n}{r} = \binom{n}{n-r}\right\}$

$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots. + \binom{n}{n}^2$

Equating coefficients of x^n,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots. + \binom{n}{n}^2$$

39 $z^2 - z + 1 + i = 0$

$$\begin{aligned}\therefore z &= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1+i)}}{2(1)}\\ &= \frac{1 \pm \sqrt{1-4-4i}}{2}\\ &= \frac{1 \pm \sqrt{-3-4i}}{2}\end{aligned}$$

Let $a + bi = \sqrt{-3-4i}$, $a, b \in \mathbb{R}$

$\therefore (a+bi)^2 = -3-4i$

$\therefore a^2 + 2abi - b^2 = -3-4i$

Equating real and imaginary parts,

$a^2 - b^2 = -3$ .... (1) $\qquad 2ab = -4$ .... (2)

From (2), $ab = -2$, and so $b = -\dfrac{2}{a}$

Substituting into (1), $a^2 - \left(\dfrac{-2}{a}\right)^2 = -3$

$$a^2 - \frac{4}{a^2} + 3 = 0$$

$$\therefore\ a^4 - 4 + 3a^2 = 0$$

$$(a^2+4)(a^2-1) = 0$$

$$\therefore\ a = \pm 1 \qquad \{a \in \mathbb{R}\}$$

$$\therefore\ b = \mp 2$$

$$\therefore\ \sqrt{3-4i} = 1 - 2i \quad \{a > 0\}$$

So, $z = \dfrac{1 \pm (1-2i)}{2} = i$ or $1 - i$

40 P_n is: "$5n^3 \quad 3n^2 \quad 2n$ is divisible by 6" for $n \subset \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5 \times 1^3 - 3 \times 1^2 - 2 \times 1 = 0$

$$= 0 \times 6$$

$\therefore$ P_1 is true.

(2) If P_k is true, then $5k^3 - 3k^2 - 2k = 6A$ where A is an integer.

$$\begin{aligned}&5(k+1)^3 - 3(k+1)^2 - 2(k+1)\\ &= 5(k^3+3k^2+3k+1) - 3(k^2+2k+1) - 2(k+1)\\ &= (5k^3 - 3k^2 - 2k) + 15k^2 + 9k\\ &= 6A + 3k(5k+3) \quad \{\text{using } P_k\}\\ &= 6A + 3(2B),\ B \in \mathbb{Z}\end{aligned}$$

$\{k(5k+3)$ is divisible by 2, since either k is divisible by 2, or k is odd $\Rightarrow 5k+3$ is divisible by 2$\}$

$= 6(A+B)$, where $A, B \in \mathbb{Z}$

So, P_{k+1} is true.

Thus P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore$ P_n is true for $n \in \mathbb{Z}^+$.

{Principle of mathematical induction}

41 **a** Since they are consecutive terms of an arithmetic sequence,

$$(k^2+5) - (3k) = 3k - (k+1)$$

$$\therefore\ k^2 - 3k + 5 = 2k - 1$$

$$\therefore\ k^2 - 5k + 6 = 0$$

$$\therefore\ (k-2)(k-3) = 0$$

$$\therefore\ k = 2 \text{ or } 3$$

b Since they are consecutive terms of a geometric sequence,

$$\frac{k^2+5}{3k} = \frac{3k}{k+1} \quad \{\text{equating ratios}\}$$

$$\therefore\ k^3 + k^2 + 5k + 5 = 9k^2$$

$$\therefore\ k^3 - 8k^2 + 5k + 5 = 0$$

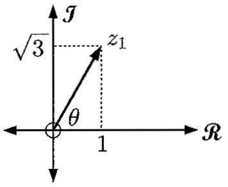
Using technology, $k \approx 1.32$ is the only solution which satisfies $0 < k < 5$.

42 Let $z = a + bi$ and $w = c + di$.

Then $(z+w)^* = (a + bi + c + di)^*$

$$\begin{aligned}&= ((a+c) + (b+d)i)^*\\ &= (a+c) - (b+d)i\\ &= (a - bi) + (c - di)\\ &= z^* + w^*\end{aligned}$$

43 **a** Let $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 + i$.



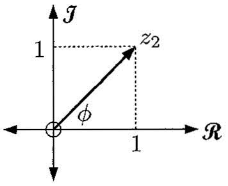
$$|z_1| = \sqrt{1^2 + (\sqrt{3})^2}$$

$$= 2$$

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ $\theta = \frac{\pi}{3}$

$\therefore$ $z_1 = 2 \operatorname{cis} \frac{\pi}{3}$



$$|z_2| = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$\cos\phi = \frac{1}{\sqrt{2}}$, $\sin\phi = \frac{1}{\sqrt{2}}$

$\therefore$ $\phi = \frac{\pi}{4}$

$\therefore$ $z_2 = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

$$\begin{aligned}\text{So } z &= \frac{1 + i\sqrt{3}}{1+i}\\ &= \frac{2 \operatorname{cis} \frac{\pi}{3}}{\sqrt{2} \operatorname{cis} \frac{\pi}{4}}\\ &= \left(\tfrac{2}{\sqrt{2}}\right) \operatorname{cis}\left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right)\\ &= \sqrt{2} \operatorname{cis} \tfrac{\pi}{12}\end{aligned}$$

b **i**

$$\begin{aligned}z^n &= \left(\sqrt{2} \operatorname{cis} \tfrac{\pi}{12}\right)^n\\ &= (\sqrt{2})^n \operatorname{cis} \tfrac{n\pi}{12}\end{aligned}$$

$\therefore$ $\arg(z^n) = \frac{n\pi}{12}$

If $z^n \in \mathbb{R}$, $\arg(z^n) = k\pi$, $k \in \mathbb{Z}$.

The smallest positive value of n which satisfies $\frac{n\pi}{12} = k\pi$ is $n = 12$.

ii If z^n is purely imaginary, $\arg(z^n) = \frac{\pi}{2} + k\pi$

$\therefore$ $\frac{n\pi}{12} = \frac{\pi}{2} + k\pi$

The smallest positive value of n which satisfies $\frac{n\pi}{12} = \frac{\pi}{2} + k\pi$ is $n = 6$.

44 $T_{r+1} = \binom{9}{r} (kx)^{9-r} \left(\dfrac{1}{\sqrt{x}}\right)^r$

$$= \binom{9}{r} k^{9-r} x^{9-r} \frac{1}{x^{\frac{r}{2}}}$$

$$= \binom{9}{r} k^{9-r} x^{9-\frac{3r}{2}}$$

For the constant term, $9 - \dfrac{3r}{2} = 0$

$$\therefore\ \frac{3r}{2} = 9$$

$$\therefore\ r = 6$$

$$T_7 = \binom{9}{6} k^3 x^0$$

$$\therefore\ 84k^3 = -10\tfrac{1}{2}$$

$$\therefore\ k^3 = -\tfrac{1}{8}$$

$$k = -\tfrac{1}{2}$$

45 **a** Writing the system in augmented form,

$$\left[\begin{array}{ccc|c} 1 & 3 & k & 2\\ k & -2 & 3 & k\\ 4 & -3 & 10 & 5\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2\\ 0 & -2-3k & 3-k^2 & -k\\ 0 & -15 & 10-4k & -3\end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - kR_1\\ R_3 \to R_3 - 4R_1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2\\ 0 & -15 & 10-4k & -3\\ 0 & -2-3k & 3-k^2 & -k\end{array}\right] \quad \begin{array}{l} R_2 = R_3\\ R_3 = R_2\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10-4k & -3 \\ 0 & 0 & 25-22k-3k^2 & 6-6k \end{array}\right]$$

$$R_3 \to 15R_3 - (2+3k)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -15 & 10-4k & -3 \\ 0 & 0 & -(3k+25)(k-1) & -6(k-1) \end{array}\right]$$

b If $k=1$, the last row is $0 \quad 0 \quad 0 \mid 0$ and the system has an infinite number of solutions.

c If $k=-\frac{25}{3}$, the last line is $0 \quad 0 \quad 0 \mid 6(1+\frac{25}{3})$ and the system has no solutions.

d If $k \neq 1$ or $-\frac{25}{3}$, the system has a unique solution.

46 **a**

$$\begin{aligned} &\log_a(5a) \\ &= \log_a 5 + \log_a a \\ &= x+1 \end{aligned}$$

b

$$\begin{aligned} &\log_a\left(\frac{a^2}{25}\right) \\ &= \log_a a^2 - \log_a 25 \\ &= 2 - 2\log_a 5 \\ &= 2-2x \end{aligned}$$

47 **a**

$$\begin{aligned} u_3 &= 20 \quad \text{and} \quad u_6 = 160 \\ \therefore\ u_1r^2 &= 20 \quad \text{and} \quad u_1r^5 = 160 \\ \therefore\ \frac{u_1r^5}{u_1r^2} &= \frac{160}{20} \\ \therefore\ r^3 &= 8 \\ \therefore\ r &= 2 \\ \text{and } u_1 2^2 &= 20 \\ \therefore\ u_1 &= 5 \end{aligned}$$

b

$$\begin{aligned} u_{10} &= 5 \times 2^9 \\ &= 2560 \end{aligned}$$

$$\sum_{n=1}^{12} u_n = S_{12}$$

$$\text{Now } S_n = \frac{u_1(r^n-1)}{r-1}$$

$$\begin{aligned} \therefore\ S_{12} &= \frac{5(2^{12}-1)}{2-1} \\ &= 20\,475 \end{aligned}$$

48 6 of the 24 upstairs seats are taken. We hence choose 18 people from the $48-(8+6)=34$ remaining passengers to sit upstairs; the rest sit downstairs.

$\therefore$ there are $\binom{34}{18} = 2\,203\,961\,430$ ways.

49 **a** Suppose $n=1$. 9^1+b is divisible by 8 if $b=7$ $\{0 < b \leqslant 9\}$

So, $b=7$.

b P_n is: "9^n+7 is divisible by 8" for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $9^1+7=16=8\times 2$

$\therefore$ P_1 is true.

(2) If P_k is true, then $9^k+7=8A$ where $A \in \mathbb{Z}^+$.

$$\begin{aligned} 9^{k+1}+7 &= 9\times 9^k+7 \\ &= 9\times(9^k+7) - 9\times 7 + 7 \\ &= 9 \times 8A - 9\times 7+7 \quad \{\text{using } P_k\} \\ &= 9\times 8A - 8\times 7 \\ &= 8(9A-7) \end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$.
{Principle of mathematical induction}

50 Let $z_1 = |z_1| \operatorname{cis}\theta$ and $z_2 = |z_2|\operatorname{cis}\phi$.

$$\begin{aligned} \text{Then } \frac{z_1}{z_2} &= \frac{|z_1|\operatorname{cis}\theta}{|z_2|\operatorname{cis}\phi} \\ &= \frac{|z_1|}{|z_2|}\operatorname{cis}(\theta-\phi) \quad \{\text{property of cis}\} \end{aligned}$$

$$\begin{aligned} \text{So } \arg\left(\frac{z_1}{z_2}\right) &= \theta - \phi \\ &= \arg(z_1) - \arg(z_2) \end{aligned}$$

51 Let u_1, u_2, and u_3 be consecutive terms of an arithmetic sequence with difference d.

$$\begin{aligned} \therefore\ u_1 = u_2 - d &\text{ and } u_3 = u_2+d. \\ \text{Now } u_1+u_2+u_3 &= 18 \\ \therefore\ u_2-d+u_2+u_2+d &= 18 \\ \therefore\ 3u_2 &= 18 \\ \therefore\ u_2 &= 6 \\ \text{Also, } u_1^2+u_2^2+u_3^2 &= 396 \\ \therefore\ (6-d)^2+6^2+(6+d)^2 &= 396 \\ \therefore\ 36-12d+d^2+36+36+12d+d^2 &= 396 \\ \therefore\ 2d^2 &= 288 \\ \therefore\ d^2 &= 144 \\ \therefore\ d &= \pm 12 \end{aligned}$$

So, the numbers are -6, 6, and 18.

52 **a**

$$\begin{aligned} \log_{10} M &= 2x-1 \\ \therefore\ 10^{\log_{10} M} &= 10^{2x-1} \\ \therefore\ M &= 10^{2x-1} \end{aligned}$$

b

$$\begin{aligned} \log_a N &= 2\log_a d - \log_a c \\ \therefore\ \log_a N &= \log_a d^2 - \log_a c \\ \therefore\ \log_a N &= \log_a\left(\frac{d^2}{c}\right) \\ \therefore\ N &= \frac{d^2}{c} \end{aligned}$$

53

$$\begin{aligned} \frac{z^2+3}{z^2-1} &= k \\ \therefore\ z^2+3 &= k(z^2-1) \\ \therefore\ z^2+3 &= kz^2-k \\ \therefore\ z^2(1-k) &= -k-3 \\ \therefore\ z^2 &= \frac{-k-3}{1-k} \end{aligned}$$

$\therefore$ the equation has imaginary roots if $\dfrac{-k-3}{1-k} < 0$

Sign diagram:

So, the equation has imaginary roots for $-3 < k < 1$.

54

$$\begin{aligned} &(x+2)(1-x)^{10} \\ &= (x+2)\left(\begin{array}{r} 1^{10} + \binom{10}{1}1^9(-x) + \ldots. + \binom{10}{4}1^6(-x)^4 \\ + \binom{10}{5}1^5(-x)^5 + \ldots. \end{array}\right) \\ &= (x+2)\left(1 - 10x + \ldots. + \binom{10}{4}x^4 - \binom{10}{5}x^5 + \ldots.\right) \end{aligned}$$

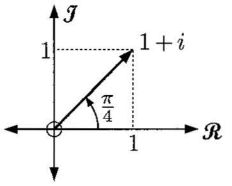
So, the terms containing x^5 are $\binom{10}{4}x^5$ and $-2\binom{10}{5}x^5$.

$\therefore$ the coefficient of x^5 is $\binom{10}{4} - 2\binom{10}{5} = -294$

55

$$\begin{aligned} z &= \frac{-1+5i}{2+3i} \\ &= \left(\frac{-1+5i}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right) \\ &= \frac{-2+3i+10i-15i^2}{4-9i^2} \\ &= \frac{13+13i}{13} \\ &= 1+i \end{aligned}$$

$$\begin{aligned} |1+i| &= \sqrt{1^2+1^2} \\ &= \sqrt{2} \end{aligned}$$



$$\begin{aligned} \arg(1+i) &= \tfrac{\pi}{4} \\ \therefore\ z &= \sqrt{2}\operatorname{cis}\left(\tfrac{\pi}{4}\right) \\ \therefore\ z^{12} &= \left[\sqrt{2}\operatorname{cis}\left(\tfrac{\pi}{4}\right)\right]^{12} \\ &= (\sqrt{2})^{12}\operatorname{cis}\left(\tfrac{12\pi}{4}\right) \quad \text{\{De Moivre's theorem\}} \\ &= 64\operatorname{cis}(\pi) \\ &= 64(\cos\pi + i\sin\pi) \\ &= -64 \end{aligned}$$

56 $2^{x-1} = 3^{2-x}$

Take the logarithm to base 10 to get:

$$\begin{aligned} \log(2^{x-1}) &= \log(3^{2-x}) \\ \therefore\ (x-1)\log 2 &= (2-x)\log 3 \\ \therefore\ x\log 2 - \log 2 &= 2\log 3 - x\log 3 \\ \therefore\ x\log 2 + x\log 3 &= 2\log 3 + \log 2 \\ \therefore\ x(\log 2 + \log 3) &= \log 3^2 + \log 2 \\ \therefore\ x\log 6 &= \log 18 \\ \therefore\ x &= \frac{\log 18}{\log 6} \\ \therefore\ x &= \log_6 18 \end{aligned}$$

So, $a = 6$ and $b = 18$.

57 $u_1 = 18$ and $d = -3$

If the series has n terms, then

$$\begin{aligned} S_n &= -210 \\ \therefore\ \frac{n}{2}(2u_1 + (n-1)d) &= -210 \\ \therefore\ \frac{n}{2}(2\times 18 + (n-1)\times(-3)) &= -210 \\ \therefore\ \frac{n}{2}(36 - 3n + 3) &= -210 \\ n(39-3n) &= -420 \\ \therefore\ 3n^2 - 39n - 420 &= 0 \\ \therefore\ 3(n^2 - 13n - 140) &= 0 \\ \therefore\ 3(n-20)(n+7) &= 0 \\ \therefore\ n &= 20 \quad \{n > 0\} \end{aligned}$$

So, there are 20 terms in the series.

58

$$\begin{aligned} |z-3| &= |z-1| \\ \therefore\ |(x+iy)-3| &= |(x+iy)-1| \\ \therefore\ |(x-3)+iy| &= |(x-1)+iy| \\ \therefore\ \sqrt{(x-3)^2+y^2} &= \sqrt{(x-1)^2+y^2} \\ \therefore\ (x-3)^2+y^2 &= (x-1)^2+y^2 \\ \therefore\ x^2-6x+9 &= x^2-2x+1 \\ \therefore\ 4x &= 8 \\ \therefore\ x &= 2 \end{aligned}$$

59 If all pairs of points defined different lines, there would be $\binom{11}{2} = 55$ lines.

There are $\binom{4}{2} = 6$ ways of choosing a pair of points from the 4 collinear points, of which we include one.

So the total number of lines is $55 - 6 + 1 = 50$.

60 P_n is: "$1 + 2\times 2^1 + 3\times 2^2 + ... + n\times 2^{n-1} = (n-1)2^n + 1$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= (1-1)2^1 + 1 = 1$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$1 + 2\times 2^1 + 3\times 2^2 + ... + k\times 2^{k-1} = (k-1)2^k + 1$

So $1 + 2\times 2^1 + 3\times 2^2 + ... + k\times 2^{k-1} + (k+1)2^k$

$$\begin{aligned} &= (k-1)2^k + 1 + (k+1)\times 2^k \quad \{\text{using } P_k\} \\ &= k\times 2^k - 2^k + 1 + k\times 2^k + 2^k \\ &= 2\times k\times 2^k + 1 \\ &= ([k+1]-1)\,2^{k+1} + 1 \end{aligned}$$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

61 **a** In augmented form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -3 & 2 & 1 \\ 3 & -4 & k & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -4 & -7 \\ 0 & 2 & k-9 & -14 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & k-1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

There is a unique solution if $k \neq 1$. Thus $k_1 = 1$.

b For $k \neq 1$, equation 3 gives $(k-1)z = 0$

$\therefore\ z = 0 \quad \{k - 1 \neq 0\}$

So in equation 2, $y - 4(0) = -7$

$\therefore\ y = -7$

and in equation 1, $x - 2(-7) + 3(0) = 4$

$\therefore\ x = -10$

$\therefore$ the unique solution is $x = -10$, $y = -7$, $z = 0$.

c When $k = 1$, there are infinitely many solutions.

Letting $z = t$, equation 2 gives $y - 4t = -7$

$\therefore\ y = 4t - 7$

Thus in equation 1, $x - 2(4t-7) + 3t = 4$

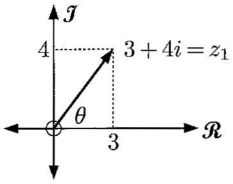
$\therefore\ x - 8t + 14 + 3t = 4$

$\therefore\ x = 5t - 10$

So, when $k = 1$ there are infinitely many solutions of the form $x = 5t - 10$, $y = 4t - 7$, $z = t$, $t \in \mathbb{R}$.

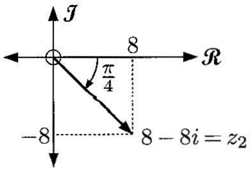
62 **a**

$$\begin{aligned} |z_1| &= \sqrt{3^2+4^2} \\ &= \sqrt{9+16} \\ &= 5 \end{aligned}$$



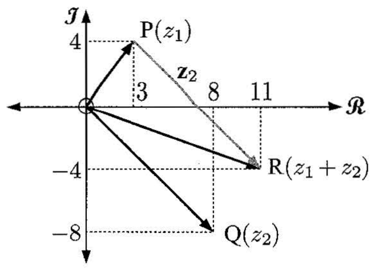
$$\begin{aligned} \therefore\ \arg(z_1) &= \arctan\left(\tfrac{4}{3}\right) \\ &\approx 0.927 \end{aligned}$$

$$\begin{aligned} |z_2| &= \sqrt{8^2+(-8)^2} \\ &= \sqrt{64+64} \\ &= 8\sqrt{2} \end{aligned}$$



$\arg(z_2) = -\frac{\pi}{4}$

b



$$\overrightarrow{\text{OR}} = \overrightarrow{\text{OP}} + \overrightarrow{\text{OQ}}$$

63 $n\binom{n-1}{r-1} = n\dfrac{(n-1)!}{(r-1)!\,(n-1-(r-1))!}$

$$= \frac{n!}{(r-1)!(n-r)!}$$

$$= r\frac{n!}{r!(n-r)!}$$

$$= r\binom{n}{r} \quad \text{for } n,\ r \in \mathbb{Z}^+,\ \ n \geqslant r.$$

64 Let $z = r_1 \operatorname{cis}\theta$ and $w = r_2 \operatorname{cis}\phi$

$$\therefore\ \frac{z}{w} = \frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\theta}$$

$$= \frac{r_1}{r_2}\operatorname{cis}(\theta - \phi) \quad \{\text{property of cis}\}$$

$$\therefore\ \left|\frac{z}{w}\right| = \frac{r_1}{r_2} = \frac{|z|}{|w|}, \quad w \neq 0$$

65 **a** $4x^2 = x^2 + 2x^2 + x^2$

$$\geqslant x^2 + 2x + 1 \quad \{\text{if } x \geqslant 1,\ x^2 \geqslant x, \text{ and } x^2 \geqslant 1\}$$

$$\geqslant (x+1)^2$$

b P_n is "$4^n \geqslant 3n^2$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 4^1 = 4$ and RHS $= 3 \times 1^2 = 3$
$4 \geqslant 3$, so P_1 is true,

(2) If P_k is true, $4^k \geqslant 3k^2$

Now, $4^{k+1} = 4 \times 4^k$

$$\geqslant 4(3k^2) \quad \{\text{using } P_k\}$$

$$\geqslant 3(4k^2)$$

$$\geqslant 3(k+1)^2 \quad \{\text{using } \mathbf{a},\ k \geqslant 1\}$$

So P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$.
{Principle of mathematical induction}

66 **a** $z^2 = z^*$

$$\therefore\ (r\operatorname{cis}\theta)^2 = (r\operatorname{cis}\theta)^*$$

$$\therefore\ r^2\operatorname{cis}2\theta = r\operatorname{cis}(-\theta) \quad \{\text{De Moivre's theorem}\}$$

$$\therefore\ r^2 = r \ \text{ and }\ \operatorname{cis}2\theta = \operatorname{cis}(-\theta)$$

$$\therefore\ \frac{\operatorname{cis}2\theta}{\operatorname{cis}(-\theta)} = 1$$

$$\therefore\ \operatorname{cis}3\theta = 1$$

b $r^2 = r$

$$\therefore\ r^2 - r = 0$$

$$\therefore\ r(r-1) = 0$$

$$\therefore\ r = 1 \quad \{r > 0\}$$

Now $\operatorname{cis}3\theta = 1$

$$\therefore\ 3\theta = 0 + 2k\pi,\ k \in \mathbb{Z}$$

$$\therefore\ \theta = \frac{2k\pi}{3},\ k \in \mathbb{Z}$$

$$\therefore\ \theta = 0,\ \tfrac{2\pi}{3},\ \tfrac{4\pi}{3} \quad \{0 \leqslant \theta \leqslant 2\pi\}$$

$$\operatorname{cis}0 = 1$$

$$\operatorname{cis}\tfrac{2\pi}{3} = \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}$$

$$= -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$$

$$\operatorname{cis}\tfrac{4\pi}{3} = \cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}$$

$$= -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

So, the non-zero solutions to $z^2 = z^*$ are
$z = 1,\ -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$ or $-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$

67 Time period = 33 months = 11 quarters
Interest rate = 8% p.a. = 2% per quarter

$$\therefore\ r = 1.02$$

$\therefore$ the amount after 11 quarters is

$$u_{12} = u_1 \times r^{11}$$

$$= 3500 \times 1.02^{11}$$

$$\approx 4351.8101$$

So, the maturing value is £4351.81.

68 Each person is either in or out of the committee, so there are $2^{12} = 4096$ possible committees.

There are $\binom{12}{1} = 12$ 1-member committees, and $\binom{12}{0} = 1$ 0-member committee.

Hence there are $4096 - (12+1) = 4083$ committees with at least two members.

69 $z = iz^*$

$$\therefore\ x + iy = i(x - iy)$$

$$= ix + y$$

$$\therefore\ x = y \quad \{\text{equating real and imaginary parts}\}$$

70 P_n is "$n^3 + 2n$ is divisible by 3" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1^3 + 2 \times 1 = 3 = 3 \times 1$
$\therefore$ P_1 is true.

(2) If P_k is true, $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$

$$(k+1)^3 + 2(k+1)$$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (k^3 + 2k) + 3k^2 + 3k + 3$$

$$= 3A + 3(k^2 + k + 1) \quad \{\text{using } P_k\}$$

$$= 3(A + k^2 + k + 1)$$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

71 The sum of the integers between 100 and 200 which are *not* a multiple of 4, is $(100 + 101 + 102 + \ldots. + 199 + 200)$
$-(100 + 104 + \ldots. + 196 + 200)$.

Now $100 + 101 + 102 + \ldots. + 199 + 200$ is an arithmetic series with $u_1 = 100$, $n = 101$, and $u_n = 200$.

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$\therefore\ S_{101} = \tfrac{101}{2}(100 + 200)$$

$$= 15\,150$$

Now $100 + 104 + \ldots. + 196 + 200$ in an arithmetic series with $u_1 = 100$, $n = 26$, and $u_n = 200$

$$\therefore\ S_{26} = \tfrac{26}{2}(100 + 200)$$

$$= 3900$$

$\therefore$ the required sum $= 15\,150 - 3900$

$$= 11\,250$$

72 **a**
$$\begin{aligned}\operatorname{cis}\theta\operatorname{cis}\phi &= e^{i\theta}e^{i\phi}\\ &= e^{i\theta+i\phi}\\ &= e^{i(\theta+\phi)}\\ &= \operatorname{cis}(\theta+\phi)\end{aligned}$$

b
$$\begin{aligned}(r\operatorname{cis}\theta)^n &= (re^{i\theta})^n\\ &= r^n(e^{i\theta})^n\\ &= r^n e^{in\theta}\\ &= r^n\operatorname{cis}n\theta\end{aligned}$$

c
$$\begin{aligned}w &= e^{i\left(\frac{2\pi}{5}\right)}\\ \therefore\ w^5 &= \left(e^{i\left(\frac{2\pi}{5}\right)}\right)^5\\ &= e^{2\pi i}\\ &= \operatorname{cis}2\pi\\ &= 1\end{aligned}$$

So $w^5 - 1 = 0$

$$\begin{aligned}\therefore\ (w-1)(1+w+w^2+w^3+w^4) &= 0\\ \therefore\ 1+w+w^2+w^3+w^4 &= 0 \quad \{w \neq 1\}\\ \therefore\ 1+w+w^2+w^3 &= -w^4\\ \therefore\ (1+w)(1+w^2) &= -w^4\end{aligned}$$

73
$$\begin{aligned}T_{r+1} &= \binom{9}{r}(\sqrt{x})^{9-r}\left(\frac{b}{x}\right)^r\\ &= \binom{9}{r}\times x^{\frac{9-r}{2}}\times b^r\times\frac{1}{x^r}\\ &= \binom{9}{r}\times b^r\times x^{\frac{9-3r}{2}}\end{aligned}$$

For the coefficient of x^{-3}, $\dfrac{9-3r}{2} = -3$
$$\therefore\ r = 5$$
$$T_6 = \binom{9}{5}b^5x^{-3}$$

Now, x^{-3} has coefficient -4032, so $\binom{9}{5}b^5 = -4032$
$$\therefore\ b^5 = -32$$
$$\therefore\ b = -2$$

74 **a** Let $u_k = 3k - 11$
$$\therefore\ u_1 = 3\times 1 - 11 = -8$$
$$\begin{aligned}u_{k+1} - u_k &= (3(k+1)-11) - (3k-11)\\ &= 3k+3-11-3k+11\\ &= 3\end{aligned}$$

Since the difference between consecutive terms is constant, u_k is an arithmetic sequence with $u_1 = -8$ and $d = 3$.

$$\begin{aligned}\therefore\ \sum_{k=1}^{n}(3k-11) &= S_n\\ \therefore\ 5536 &= \frac{n}{2}(2u_1 + (n-1)d)\\ \therefore\ 5536 &= \frac{n}{2}(2\times(-8) + (n-1)\times 3)\\ \therefore\ 11\,072 &= n(-16+3n-3)\end{aligned}$$
$$\therefore\ 3n^2 - 19n - 11\,072 = 0$$
$$\therefore\ (3n+173)(n-64) = 0$$
$$n = 64 \text{ or } -\tfrac{173}{3}$$

Since n must be an integer, $n = 64$.

b Let $u_k = \left(\dfrac{y}{5}\right)^{k-1}$
$$\therefore\ u_1 = \left(\frac{y}{5}\right)^{1-1} = \left(\frac{y}{5}\right)^0 = 1$$

and $\dfrac{u_{k+1}}{u_k} = \dfrac{\left(\frac{y}{5}\right)^k}{\left(\frac{y}{5}\right)^{k-1}} = \dfrac{y}{5}$ for all $k \in \mathbb{Z}^+$

Since the ratio between consecutive terms is constant, u_k is a geometric sequence with $u_1 = 1$ and $r = \dfrac{y}{5}$.

$$\sum_{k=1}^{\infty}\left(\frac{y}{5}\right)^{k-1} = S = \frac{u_1}{1-r}$$
$$\therefore\ 5 = \frac{1}{1-\frac{y}{5}}$$
$$\therefore\ \left(1-\frac{y}{5}\right)\times 5 = 1$$
$$\therefore\ 5 - y = 1$$
$$\therefore\ y = 4$$

75 If the mathematics exams are consecutive, then we can treat them as one subject, giving 7 exams to order.

This pair of mathematics subjects has 2! orderings, so there are 7! × 2! schedules that cannot be used.

There are 8! total orderings.

So the teacher can choose from $8! - (7! \times 2!) = 30\,240$ schedules.

76 **a**
$$\begin{aligned}w &= e^{i\left(\frac{2\pi}{3}\right)}\\ \therefore\ w^3 &= \left(e^{i\left(\frac{2\pi}{3}\right)}\right)^3\\ &= e^{2\pi i}\\ &= \operatorname{cis}2\pi\\ &= 1\\ \therefore\ w^3 - 1 &= 0\\ \therefore\ (w-1)(1+w+w^2) &= 0\\ \therefore\ 1+w+w^2 &= 0 \quad \{w\neq 1\}\end{aligned}$$

b **i**
$$\begin{aligned}w^7 &= w^3w^3w\\ &= w\end{aligned}$$

ii
$$\begin{aligned}w^3 &= 1\\ \therefore\ w^2 &= \frac{1}{w}\\ \therefore\ w^{-1} &= w^2\end{aligned}$$

iii
$$\begin{aligned}(1-w)^2 &= 1 - 2w + w^2\\ &= 1 + w + w^2 - 3w\\ &= -3w\end{aligned}$$

iv
$$\begin{aligned}\frac{1}{(1+w)^2} &= \frac{1}{1+2w+w^2}\\ &= \frac{1}{1+w+w^2+w}\\ &= \frac{1}{w}\\ &= w^{-1}\\ &= w^2 \quad \{\text{part } \mathbf{ii}\}\end{aligned}$$

v
$$\begin{aligned}\frac{1+w^2}{1+w} &= \frac{1+w+w^2-w}{1+w+w^2-w^2}\\ &= \frac{-w}{-w^2}\\ &= w^{-1}\\ &= w^2\end{aligned}$$

77 P_n is: "$3^n > n^2 + n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 3^1 = 3$ and RHS $= 1^2 + 1 = 2$

$\therefore$ P_1 is true.

(2) If P_k is true, $3^k > k^2 + k$
$$\begin{aligned}3^{k+1} &= 3\times 3^k\\ &> 3(k^2+k) \qquad \{\text{using } P_k\}\\ &> 3k^2 + 3k\\ &> 2k^2 + k^2 + 3k\\ &> 2 + k^2 + 3k \quad \{k \geqslant 1\}\\ &> k^2 + 2k + 1 + k + 1\\ &> (k+1)^2 + (k+1)\end{aligned}$$

So, P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

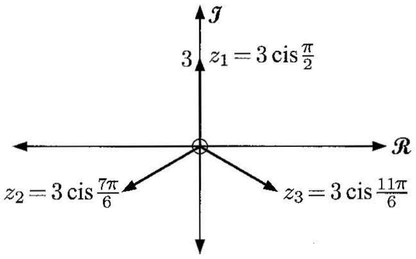
78 **a** Let $z^3 = -27i$

$\therefore\ z^3 = 27\operatorname{cis}\left(\frac{3\pi}{2}\right)$

$\therefore\ z^3 = 27\operatorname{cis}\left(\frac{3\pi}{2} + k2\pi\right),\ k \in \mathbb{Z}$

$\therefore\ z = 3\operatorname{cis}\left(\frac{\pi}{2} + \frac{k2\pi}{3}\right)$ {De Moivre's theorem}

$\therefore\ z = 3\operatorname{cis}\frac{\pi}{2},\ 3\operatorname{cis}\frac{7\pi}{6},\ 3\operatorname{cis}\frac{11\pi}{6}$ $\{k = 0,\ 1,\ 2\}$



b Let $z_1 = 3\operatorname{cis}\frac{\pi}{2},\ z_2 = 3\operatorname{cis}\frac{7\pi}{6}$

$z_3 = 3\operatorname{cis}\frac{11\pi}{6}$

$$\begin{aligned} z_2z_3 &= 3\operatorname{cis}\tfrac{7\pi}{6} \times 3\operatorname{cis}\tfrac{11\pi}{6} \\ &= 9\operatorname{cis}\left(\tfrac{18\pi}{6}\right) \\ &= -9 \\ z_1^2 &= \left(3\operatorname{cis}\tfrac{\pi}{2}\right)^2 \\ &= 9\operatorname{cis}\pi \\ &= -9 \text{ and so } z_2z_3 = z_1^2 \end{aligned}$$

c $z_1z_2z_3 = z_1(z_1^2)$ {from **b**}

$$\begin{aligned} &= z_1^3 \\ &= -27i \end{aligned}$$

79 **a** $\binom{16}{3} = 560$ triangles

b There are 5 possible points on the circle and $\binom{11}{2}$ possible points within the circle.

$\therefore$ there are $5 \times \binom{11}{2} = 275$ triangles.

c There are $\binom{5}{3} = 10$ triangles with all vertices on the circle. Hence there are $560 - 10 = 550$ triangles with at least one of the vertices within the circle.

80 **a**

$$\begin{aligned} & z^n + \frac{1}{z^n} \\ &= z^n + z^{-n} \\ &= (\operatorname{cis}\theta)^n + (\operatorname{cis}\theta)^{-n} \\ &= \operatorname{cis}(n\theta) + \operatorname{cis}(-n\theta) \quad \text{\{De Moivre's theorem\}} \\ &= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta) \\ &= 2\cos(n\theta) \end{aligned}$$

b $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$\therefore\ \left(z + \frac{1}{z}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$\therefore\ (2\cos\theta)^4 = (2\cos 4\theta) + 4(2\cos 2\theta) + 6$

$\therefore\ 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore\ \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

81 **a** $(2, 4)$, $(2, -6)$, and $(-1, 3)$ lie on a circle with equation $x^2 + y^2 + ax + by + c = 0$

$\therefore\ 2^2 + 4^2 + a(2) + b(4) + c = 0$

$\therefore\ 2^2 + (-6)^2 + a(2) + b(-6) + c = 0$

$\therefore\ (-1)^2 + 3^2 + a(-1) + b(3) + c = 0$

$\therefore\ 2a + 4b + c = -20$

$\therefore\ 2a - 6b + c = -40$

$\therefore\ -a + 3b + c = -10$

b In augmented matrix form, the system is

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & -10 \\ 2 & 4 & 1 & -20 \\ 2 & -6 & 1 & -40 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & 10 \\ 0 & 10 & 3 & -40 \\ 0 & 0 & 3 & -60 \end{array}\right] \begin{array}{l} R_1 \to -R_1 \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$\therefore\ 3c = -60$

$\therefore\ c = -20$

Using row 2, $10b + 3(-20) = -40$

$\therefore\ 10b = 20$

$\therefore\ b = 2$

Using row 1, $a - 3(2) - (-20) = 10$

$\therefore\ a = -4$

So the equation of the circle is

$x^2 + y^2 - 4x + 2y - 20 = 0$

$\therefore\ (x-2)^2 - 4 + (y+1)^2 - 1 = 20$

$\therefore\ (x-2)^2 + (y+1)^2 = 25$

Thus the circle's centre is at $(2, -1)$.

82 Rate per month $= \frac{5}{12}\% = \frac{0.05}{12}$ $\therefore\ R = \frac{0.05}{12}$

a After 3 months, her amount is

$60(1+R)^3 + 60(1+R)^2 + 60(1+R)$

which is geometric with

$u_1 = 60(1+R)$ and $r = 1 + R$.

So, $S_3 = 60(1+R)\left[\dfrac{(1+R)^3 - 1}{1 + R - 1}\right]$

$= 60\left(1 + \dfrac{0.05}{12}\right)\left[\dfrac{\left(1 + \frac{0.05}{12}\right)^3 - 1}{\frac{0.05}{12}}\right]$

$= \$181.50$

b After k years or $12k$ months,

$S_{12k} = 60\left(1 + \dfrac{0.05}{12}\right)\left[\dfrac{\left(1 + \frac{0.05}{12}\right)^{12k} - 1}{\frac{0.05}{12}}\right]$

$\therefore\ S_{12k} \approx 14\,460\left([1.004\,166]^{12k} - 1\right)$

c After 20 years, $k = 20$

$S_{240} \approx 14\,460\left([1.004\,166\,6]^{240} - 1\right)$

$\approx \$24\,765$ {nearest \$}

83 **a** $\dfrac{1}{u_1} + \dfrac{1}{u_2} + \dfrac{1}{u_3} + \dots + \dfrac{1}{u_n}$

$= \dfrac{1}{u_1} + \dfrac{1}{u_1r} + \dfrac{1}{u_1r^2} + \dots + \dfrac{1}{u_1r^{n-1}}$

The series is geometric with first term $\dfrac{1}{u_1}$ and common ratio $\dfrac{1}{r}$.

$$\begin{aligned} \therefore\ \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots + \frac{1}{u_n} &= \frac{1}{u_1}\left(\frac{(\frac{1}{r})^n - 1}{\frac{1}{r} - 1}\right) \\ &= \frac{1}{u_1}\left(\frac{1 - r^n}{r^n(\frac{1}{r} - 1)}\right) \\ &= \frac{1}{u_1}\left(\frac{1 - r^n}{r^{n-1} - r^n}\right) \end{aligned}$$

b $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{3072}$

is the sum of the reciprocals of $3, 6, 12, \dots, 3072$

which is geometric with $u_1 = 3$, $r = 2$.

$u_n = u_1r^{n-1}$ $\therefore\ 3072 = 3 \times 2^{n-1}$

$\therefore\ 2^{n-1} = 1024$

$\therefore\ 2^{n-1} = 2^{10}$ and so $n = 11$

$\therefore\ S_{11} = \frac{1}{3}\left(\dfrac{1 - 2^{11}}{2^{10} - 2^{11}}\right) \approx 0.666$

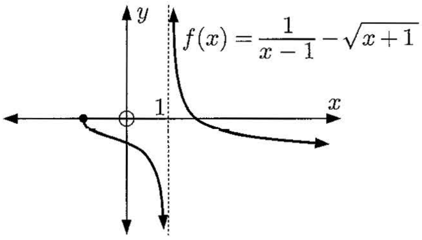
SOLUTIONS TO TOPIC 2 (FUNCTIONS AND EQUATIONS)

1 A *function* is a relation in which no two different ordered pairs have the same x-coordinate. A graph of a relation is a function if all possible vertical lines on the graph cut the relation no more than once.

2 **a** Domain $= \{x \mid x > 2\}$, Range $= \{y \mid y \in \mathbb{R}\}$

b Domain $= \{x \mid x \geqslant -1, x \neq 1\}$

Range $= \{y \mid y \in \mathbb{R}\}$



3 **a**
$$\begin{aligned} f(g(x)) &= f(4-x) \\ &= 3(4-x)+1 \\ &= -3x+13 \end{aligned}$$

b
$$\begin{aligned} (g \circ f)(x) &= g(3x+1) \\ &= 4-(3x+1) \\ &= 3-3x \end{aligned}$$

So
$$\begin{aligned} (g \circ f)(-4) &= 3-3(-4) \\ &= 15 \end{aligned}$$

c f is $y = 3x+1$, so f^{-1} is $x = 3y+1$

$$\begin{aligned} \therefore\ y &= \frac{x-1}{3} \\ \therefore\ f^{-1}(x) &= \frac{x-1}{3} \\ \text{so } f^{-1}(\tfrac{1}{2}) &= \frac{\frac{1}{2}-1}{3} \\ &= -\tfrac{1}{6} \end{aligned}$$

4 **a** For a reflection in the y-axis, we replace x with $-x$

$\therefore$ the equation is $y = -\frac{2}{x}$.

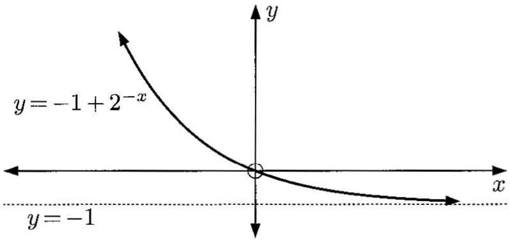
b The translated image is $y - 2 = \frac{2}{x-(-1)}$

which is $y = \frac{2}{x+1} + 2$.

c For a horizontal stretch with scale factor 3, we replace x with $\frac{x}{3}$

$\therefore$ the equation is $y = \frac{2}{\frac{x}{3}}$ which is $y = \frac{6}{x}$.

5



6 **a** The axis of symmetry is $x = \frac{-4}{2 \times 2} = -1$

b $f(-1) = 2(-1)^2 + 4(-1) - 20 = -22$

$\therefore$ the turning point is at $(-1, -22)$

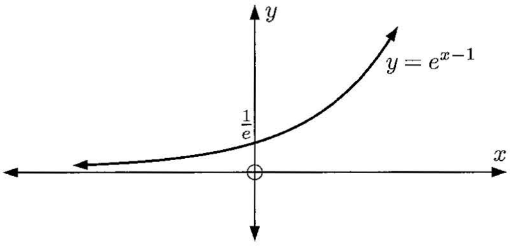
c **i** $y = 2(x+1)^2 - 22$

ii f has zeros at

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times (-20)}}{2 \times 2} \\ &= \frac{-4 \pm \sqrt{176}}{4} \\ &= -1 \pm \sqrt{11} \end{aligned}$$

So $y = 2(x+1+\sqrt{11})(x+1-\sqrt{11})$.

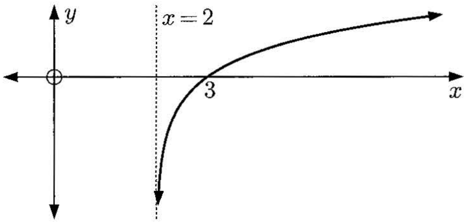
7 **a**



b Domain $= \{x \mid x \in \mathbb{R}\}$

Range $= \{y \mid y > 0\}$

8 **a**

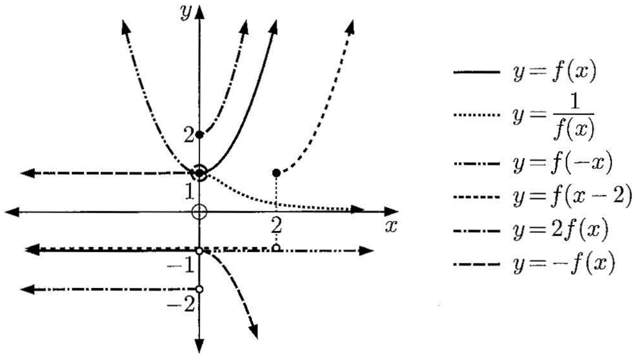


b Domain $= \{x \mid x > 2\}$

Range $= \{y \mid y \in \mathbb{R}\}$

$x = 2$ is a vertical asymptote.

9



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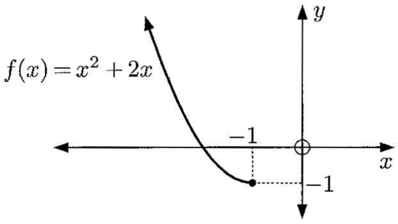


f is $y = x^2 + 2x$, $x \in \,]-\infty, -1]$

so f^{-1} is $x = y^2 + 2y$, $y \in \,]-\infty, -1]$

$\therefore\ x + 1 = y^2 + 2y + 1$

$\therefore\ x + 1 = (y+1)^2$

$\therefore\ \pm\sqrt{x+1} = y + 1$

$y = -\sqrt{x+1} - 1 \quad \{\text{as } y \leqslant -1\}$

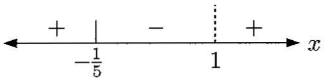
so $f^{-1}(x) = -\sqrt{x+1} - 1$

Any point of $f(x)$ which lies on the line $y = x$ will be an invariant point.

$\therefore\ (-1, -1)$ is an invariant point.

11 **a**

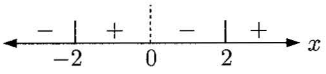
$$\begin{aligned} \frac{2x}{x-1} &\leqslant \frac{1}{3} \\ \therefore\ \frac{2x}{x-1} - \frac{1}{3} &\leqslant 0 \\ \therefore\ \frac{2x}{x-1}\left(\frac{3}{3}\right) - \frac{1}{3}\left(\frac{x-1}{x-1}\right) &\leqslant 0 \\ \frac{6x-(x-1)}{3(x-1)} &\leqslant 0 \\ \frac{5x+1}{3(x-1)} &\leqslant 0 \end{aligned}$$



$\therefore\ x \in [-\frac{1}{5}, 1[$

b

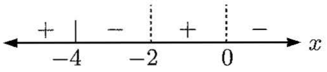
$$\begin{aligned} x &\geqslant \frac{4}{x} \\ \therefore\ x - \frac{4}{x} &\geqslant 0 \\ \therefore\ \frac{x^2-4}{x} &\geqslant 0 \\ \therefore\ \frac{(x+2)(x-2)}{x} &\geqslant 0 \end{aligned}$$



$\therefore\ x \in [-2, 0[$ or $[2, \infty[$

c

$$\begin{aligned} \frac{1}{x+2} &> \frac{2}{x} \\ \therefore\ \frac{1}{x+2} - \frac{2}{x} &> 0 \\ \therefore\ \frac{x-2(x+2)}{x(x+2)} &> 0 \\ \therefore\ \frac{-x-4}{x(x+2)} &> 0 \end{aligned}$$

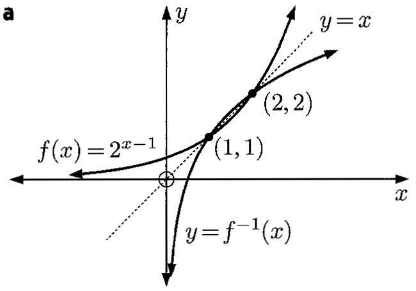


$\therefore\ x \in]-\infty, -4[$ or $]-2, 0[$

12 $p + q = -\frac{b}{a} = -\frac{1}{3}$ and $pq = \frac{c}{a} = \frac{-1}{3}$

$$\begin{aligned} p^2 + q^2 &= (p+q)^2 - 2pq \\ &= \left(-\tfrac{1}{3}\right)^2 - 2\left(-\tfrac{1}{3}\right) \\ &= \tfrac{7}{9} \end{aligned}$$

13 **a**



b

$f(x)$ is $y = 2^{x-1}$

so $f^{-1}(x)$ is $x = 2^{y-1}$

$$\begin{aligned} \therefore\ \log x &= (y-1)\log 2 \\ \therefore\ \frac{\log x}{\log 2} &= y - 1 \\ \therefore\ y &= \log_2 x + 1 \\ \therefore\ f^{-1}(x) &= \log_2 x + 1 \end{aligned}$$

14
$$\begin{aligned} (h \circ f)(x) &= h(x^2+2) \\ &= 3 - 2(x^2+2) \\ &= -2x^2 - 1 \end{aligned}$$

15 Let $P(x) = 2x^5 - x^3 + 4x - 1$

$$\begin{aligned} P(-1) &= 2(-1)^5 - (-1)^3 + 4(-1) - 1 \\ &= -6 \end{aligned}$$

$\therefore$ by the Remainder theorem, the remainder when $P(x)$ is divided by $x+1$ is -6.

16 $y = kx - 2$ meets $y = 3x^2 + x + 1$

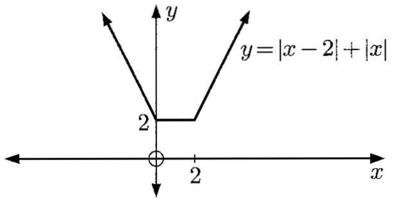
where $kx - 2 = 3x^2 + x + 1$

$\therefore\ 3x^2 + (1-k)x + 3 = 0$

Now this quadratic has $\Delta = 0$ since the graphs touch.

$$\begin{aligned} \therefore\ (1-k)^2 - 4 \times 3 \times 3 &= 0 \\ \therefore\ (1-k)^2 &= 36 \\ \therefore\ 1 - k &= \pm 6 \\ \therefore\ k &= 1 \pm 6 \\ \therefore\ k &= -5 \text{ or } 7 \end{aligned}$$

17 **a** $|x-2| + |x| = \begin{cases} 2 - 2x, & x < 0 \\ 2, & 0 \leqslant x < 2 \\ 2x - 2, & x \geqslant 2 \end{cases}$



b **i** $|x-2| + |x| = 2$

$\therefore\ x \in [0, 2]$

ii $|x-2| + |x| \geqslant 3$

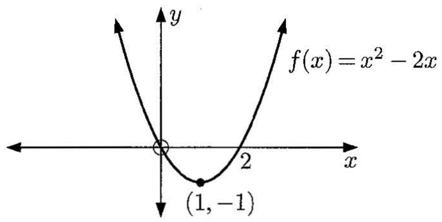
$\therefore\ x \in]-\infty, -\frac{1}{2}]$ or $[\frac{5}{2}, \infty[$

18 $f(x)$ is $y = e^{2x+1}$

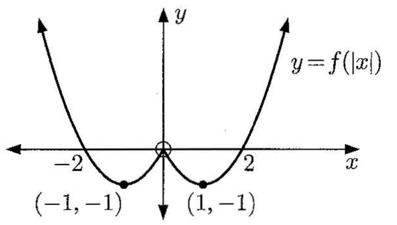
so $f^{-1}(x)$ is $x = e^{2y+1}$

$$\begin{aligned} \therefore\ \ln x &= \ln e^{2y+1} \\ \therefore\ \ln x &= 2y + 1 \\ \therefore\ y &= \frac{\ln x - 1}{2} \\ \text{so } f^{-1}(x) &= \frac{\ln x - 1}{2} \\ \therefore\ f^{-1}(7) &= \frac{\ln 7 - 1}{2} \\ &\approx 0.473 \end{aligned}$$

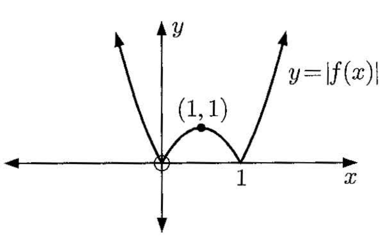
19 **a**



b **i**



ii



20 Let $P(x) = 2x^3 + 4x^2 + dx - 6$.

If $x + 2$ is a factor, then $P(-2) = 0$ {Factor theorem}

$$\begin{aligned}\therefore\ 2(-2)^3 + 4(-2)^2 + d(-2) - 6 &= 0\\ \therefore\ -16 + 16 - 2d - 6 &= 0\\ \therefore\ d &= -3\end{aligned}$$

21 **a** A root of 0 indicates a factor x.

A root of $\frac{1}{2}$ indicates a factor $2x - 1$.

So, quadratic equations with roots 0 and $\frac{1}{2}$ have the form $ax(2x - 1) = 0$ where $a \neq 0$.

b A root of $\frac{2}{3}$ indicates a factor $3x - 2$.

A root of $-\frac{1}{4}$ indicates a factor $4x + 1$.

So, quadratic equations with roots $\frac{2}{3}$ and $-\frac{1}{4}$ have the form $a(3x - 2)(4x + 1) = 0$ where $a \neq 0$.

c Sum of roots $= -4$

$$\begin{aligned}\text{Product of roots } &= (-2 + \sqrt{2})(-2 - \sqrt{2})\\ &= 4 - 2\\ &= 2\end{aligned}$$

So, quadratic equations with roots $-2 \pm \sqrt{2}$ have the form $a(x^2 + 4x + 2) = 0$ where $a \neq 0$.

d Sum of roots $= -2$

$$\begin{aligned}\text{Product of roots } &= (-1 + i\sqrt{3})(-1 - i\sqrt{3})\\ &= 1 + 3\\ &= 4\end{aligned}$$

So, quadratic equations with roots $-1 \pm i\sqrt{3}$ have the form $a(x^2 + 2x + 4) = 0$ where $a \neq 0$.

22

$$\begin{aligned}P(-2) &= 47 \quad \{\text{Remainder theorem}\}\\ \therefore\ 2(-2)^n - 10(-2) - 5 &= 47\\ \therefore\ 2(-2)^n + 15 &= 47\\ \therefore\ 2(-2)^n &= 32\\ \therefore\ (-2)^n &= 16\\ \therefore\ n &= 4\end{aligned}$$

23

$$\begin{aligned}9x^4 + 4 &= (3x^2)^2 - (2i)^2\\ &= (3x^2 + 2i)(3x^2 - 2i) \quad \{\text{Difference of two squares}\}\end{aligned}$$

24 $g(x)$ is $y = \log_2(2x - 1)$

$\therefore\ g^{-1}(x)$ is $x = \log_2(2y - 1)$

$$\begin{aligned}\therefore\ 2^x &= 2y - 1\\ \therefore\ y &= \frac{2^x + 1}{2}\\ \text{So, } g^{-1}(x) &= \frac{2^x + 1}{2}\\ \therefore\ g^{-1}(-6) &= \frac{2^{-6} + 1}{2}\\ &= \tfrac{65}{128}\end{aligned}$$

25 Since $mx^2 + (m - 2)x + m = 0$ has a repeated root, $\Delta = 0$.

$$\begin{aligned}\therefore\ (m - 2)^2 - 4 \times m \times m &= 0\\ \therefore\ -3m^2 - 4m + 4 &= 0\\ \therefore\ -(3m - 2)(m + 2) &= 0\\ \therefore\ m &= \tfrac{2}{3} \text{ or } -2\end{aligned}$$

26 **a** **i** If $f(x) = x - \dfrac{1}{x}$

$$\begin{aligned}\text{then } f(-x) &= -x - \frac{1}{(-x)}\\ &= -\left(x - \frac{1}{x}\right)\\ &= -f(x)\end{aligned}$$

$\therefore\ y = x - \dfrac{1}{x}$ is an odd function.

ii If $f(x) = \cos 2x$

$$\begin{aligned}\text{then } f(-x) &= \cos(-2x)\\ &= \cos 2x\\ &= f(x)\end{aligned}$$

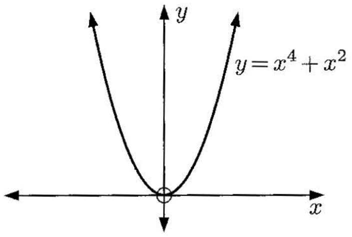
$\therefore\ y = \cos 2x$ is an even function.

b **i** A function is even if $f(-x) = f(x)$ for all x in the domain of the function.

$\therefore$ for any value in the range of f, there exists at least two corresponding values of x in the domain (except in the trivial case $f(x) = k$ with domain $x = 0$)

$\therefore$ the function fails the horizontal line test, and does not have an inverse.

ii



If $f(x) = x^4 + x^2$

$$\begin{aligned}\text{then } f(-x) &= (-x)^4 + (-x)^2\\ &= x^4 + x^2\\ &= f(x)\end{aligned}$$

$\therefore\ f(x)$ is even.

From the graph, we observe that the function is strictly increasing for $x \geqslant 0$.

$\therefore$ we can choose the domain restriction $x \geqslant 0$ for the function to have an inverse.

27 $(-2)^2 + b(-2) + (b - 2) = 0$ {-2 is a solution}

$$\begin{aligned}\therefore\ 4 - 2b + b - 2 &= 0\\ \therefore\ b &= 2\end{aligned}$$

$\therefore$ the equation is $x^2 + 2x = 0$

$\therefore\ x(x + 2) = 0$

$\therefore\ x = 0$ is the other root.

28

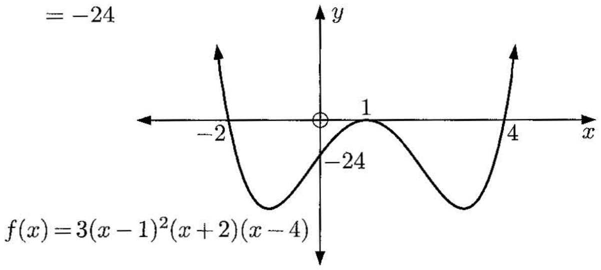
$$\begin{array}{rrrrr}
 & & & 2x & -\ 3\\ \cline{2-5}
x^2 - 4x + 4 & \multicolumn{1}{|r}{2x^3} & -11x^2 & +21x & -\ 8\\
 & -(2x^3 & -\ 8x^2 & +\ 8x) & \\ \cline{2-4}
 & & -\ 3x^2 & +13x & -\ 8\\
 & & -(-\ 3x^2 & +12x & -12)\\ \cline{3-5}
 & & & x & +\ 4
\end{array}$$

So $2x^3 - 11x^2 + 21x - 8 = (x - 2)^2(2x - 3) + x + 4$

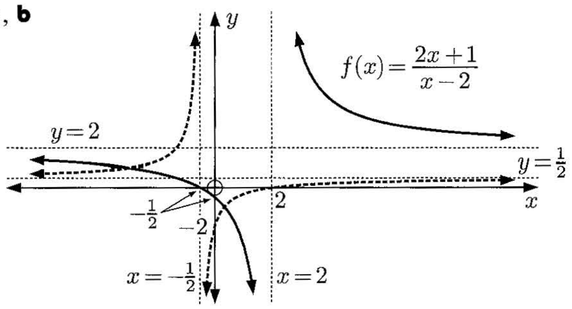
$\therefore\ 2x^3 - 11x^2 + 21x - 8 = (x - 2)^2(2x - 3) + (x - 2) + 6$

$$\therefore\ \frac{2x^3 - 11x^2 + 21x - 8}{(x - 2)^2} = 2x - 3 + \frac{1}{x - 2} + \frac{6}{(x - 2)^2}$$

29 $f(0) = 3(-1)^2(2)(-4)$
$= -24$



30 **a, b**



c Invariant points require $f(x) = \dfrac{1}{f(x)}$

$\therefore \quad \dfrac{2x+1}{x-2} = \dfrac{x-2}{2x+1}$

$\therefore \quad 4x^2 + 4x + 1 = x^2 - 4x + 4$

$\therefore \quad 3x^2 + 8x - 3 = 0$

$\therefore \quad (3x - 1)(x + 3) = 0$

$\therefore \quad x = \frac{1}{3}, -3$

The invariant points are $(-3, 1)$ and $(\frac{1}{3}, -1)$.

31 $1 + i$ is a zero of $x^3 + ax^2 - 4ax + 6$

$\therefore \quad (1+i)^3 + a(1+i)^2 - 4a(1+i) + 6 = 0$

$\therefore \quad 1 + 3i + 3i^2 + i^3 + a(1 + 2i + i^2) - 4a - 4ai + 6 = 0$

$\therefore \quad 1 + 3i - 3 - i + a + 2ai - a - 4a - 4ai + 6 = 0$

$\therefore \quad (4 - 4a) + i(2 - 2a) = 0$

Equating real and imaginary parts,

$4 - 4a = 0$ and $2 - 2a = 0$

$\therefore \quad a = 1$

32 **a**

$$\begin{array}{r|rrrrr}
 & & x^3 & -2x^2 & +7x & -13 \\ \hline
x+2 & x^4 & +0x^3 & +3x^2 & +x & +0 \\
 & -(x^4 & +2x^3) & & & \\ \cline{2-3}
 & & -2x^3 & +3x^2 & & \\
 & & -(-2x^3 & -4x^2) & & \\ \cline{3-4}
 & & & 7x^2 & +x & \\
 & & & -(7x^2 & +14x) & \\ \cline{4-5}
 & & & & -13x & +0 \\
 & & & & -(-13x & -26) \\ \cline{5-6}
 & & & & & +26
\end{array}$$

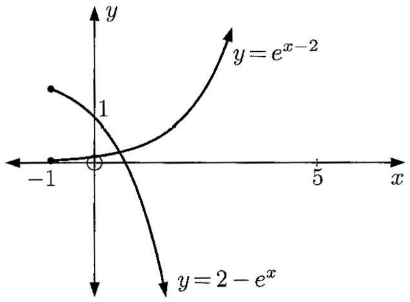
$\therefore$ quotient is $x^3 - 2x^2 + 7x - 13$ and remainder is 26

b $(x-1)^2 = x^2 - 2x + 1$

$$\begin{array}{r|rrrrr}
 & & & x^2 & +2x & +6 \\ \hline
x^2-2x+1 & x^4 & +0x^3 & +3x^2 & +x & +0 \\
 & -(x^4 & -2x^3 & +x^2) & & \\ \cline{2-4}
 & & 2x^3 & +2x^2 & +x & \\
 & & -(2x^3 & -4x^2 & +2x) & \\ \cline{3-5}
 & & & 6x^2 & -x & +0 \\
 & & & -(6x^2 & -12x & +6) \\ \cline{4-6}
 & & & & 11x & -6
\end{array}$$

$\therefore$ quotient is $x^2 + 2x + 6$ and remainder is $11x - 6$

33



Using technology, the point of intersection is $\approx (0.566, 0.238)$.

34 $y = 3x^2 + 2x$

Vertical stretch: $y = 2(3x^2 + 2x)$

$\therefore \quad y = 6x^2 + 4x$

Translation by 3 horizontally and -1 vertically:

$y = 6(x-3)^2 + 4(x-3) - 1$
$= 6(x^2 - 6x + 9) + 4x - 12 - 1$
$= 6x^2 - 32x + 41$

35 sum of roots $= \alpha + \beta = -m$

product of roots $= \alpha\beta = 1$

Now $\alpha\beta = \dfrac{1}{\alpha} + \dfrac{1}{\beta}$

$\therefore \quad \alpha\beta = \dfrac{\alpha + \beta}{\alpha\beta}$

$\therefore \quad 1 = \dfrac{-m}{1}$

$\therefore \quad m = -1$

36 $y = dx+2$ meets $y = x^2+3x+3$ when $dx+2 = x^2+3x+3$

$\therefore \quad x^2 + (3-d)x + 1 = 0$

Since the graphs meet twice, this equation must have a positive discriminant.

$\therefore \quad (3-d)^2 - 4 \times 1 \times 1 > 0$

$\therefore \quad 9 - 6d + d^2 - 4 > 0$

$\therefore \quad d^2 - 6d + 5 > 0$

$\therefore \quad (d-5)(d-1) > 0$

Sign diagram: $\xleftarrow{\quad + \quad} 1 \xrightarrow{\quad - \quad} 5 \xrightarrow{\quad + \quad} d$

So $d < 1$ or $d > 5$.

37 $P(1) = 0$ {Factor theorem}

$\therefore \quad 2 + a + b - 3 = 0$

$\therefore \quad a = 1 - b$ .... (1)

Also, $P(-3) = 0$ {Factor theorem}

$\therefore \quad -54 + 9a - 3b - 3 = 0$

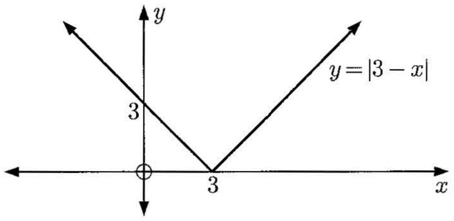
$\therefore \quad 9a - 3b = 57$

$$\begin{aligned}\text{Using (1),}\quad 9(1-b)-3b&=57\\ \therefore\ -12b&=48\\ \therefore\ b&=-4\\ \therefore\ a&=5\end{aligned}$$

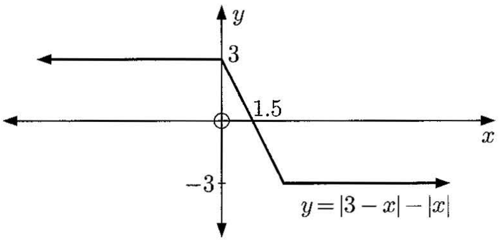
So $P(x) = 2x^3+5x^2-4x-3$
$= (x+3)(x-1)(2x+1)$ {inspection}

$\therefore$ the three zeros are -3, 1 and $-\frac{1}{2}$.

38 **a**



b $|3-x|-|x| = \begin{cases} 3, & x<0 \\ 3-2x, & 0\leqslant x<3 \\ -3, & x\geqslant 3 \end{cases}$



39 If $a+2i$ is a root, so is its complex conjugate $a-2i$ $\{a, b\in\mathbb{R}\}$.

$\therefore$ product of roots $= (a+2i)(a-2i)$
$= a^2+4$

$\therefore\ a^2+4 = a+6$
$\therefore\ a^2-a-2=0$
$\therefore\ (a-2)(a+1)=0$
$\therefore\ a=-1$ or 2

Also, sum of roots $= 2a$
$\therefore\ b=-2a$
$\therefore\ a=-1, b=2$ or $a=2, b=-4$

40 **a** $y = 4-\ln(x-2)$ has the vertical asymptote $x=2$.

b $y = \dfrac{3+x}{2x-1}$ has the vertical asymptote $x=\frac{1}{2}$.

As $x\to\pm\infty$, $y\to\frac{1}{2}$

$\therefore$ the graph also has the horizontal asymptote $y=\frac{1}{2}$.

c $y=2e^{x-4}$ has the horizontal asymptote $y=0$.

41 $(-b)^3+b(-b)^2+a(-b)+ab$
$= -b^3+b^3-ab+ab$
$=0$

$\therefore\ -b$ is a zero of $x^3+bx^2+ax+ab$

$\therefore\ x^3+bx^2+ax+ab$
$= (x+b)(x^2+a)$
$= (x+b)(x+i\sqrt{a})(x-i\sqrt{a})$

42 $(x+1)$ and $(x-2)^2$ must be factors, given the x-intercepts of $P(x)$.

$\therefore\ P(x) = (ax+b)(x+1)(x-2)^2$

Now $P(0)=56$

$\therefore\ (a\times 0+b)(1)(-2)^2=56$
$\therefore\ 4b=56$
$\therefore\ b=14$

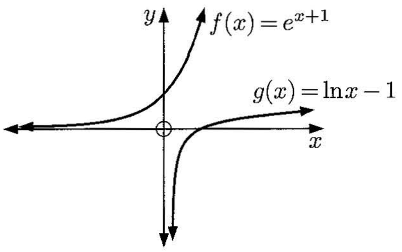
Also, $P(1)=20$

$\therefore\ (a\times 1+14)(2)(-1)^2=20$
$(a+14)\times 2=20$
$\therefore\ a=-4$

$$\begin{aligned}\therefore\ P(x) &= (-4x+14)(x+1)(x-2)^2\\ &= (-4x+14)(x+1)(x^2-4x+4)\\ &= (-4x+14)(x^3-3x^2+4)\\ &= -4x^4+26x^3-42x^2-16x+56\end{aligned}$$

43 **a**
$$\begin{aligned}(f\circ g)(x) &= f(g(x)) & (g\circ f)(x) &= g(f(x))\\ &= e^{\ln x-1+1} & &= \ln(e^{x+1})-1\\ &= e^{\ln x} & &= x+1-1\\ &= x & &= x\end{aligned}$$

b



c f and g are the inverses of one another, so $f^{-1}=g$ and $g^{-1}=f$.

44 Since the graph does not cut the x-axis, $\Delta<0$.

$\therefore\ (m-1)^2-4\times m\times 2<0$
$\therefore\ m^2-2m+1-8m<0$
$\therefore\ m^2-10m+1<0$

Now $m^2-10m+1=0$

when $m = \dfrac{10\pm\sqrt{(-10)^2-4\times1\times1}}{2\times1}$
$= 5\pm2\sqrt{6}$

So, $5-2\sqrt{6}<m<5+2\sqrt{6}$

45 $P(x)$ divided by $(x-1)(x-2)$ gives a remainder of $2x+3$.

$\therefore\ P(x) = (x-1)(x-2)Q(x)+2x+3$
for some polynomial $Q(x)$.

$\therefore\ P(1) = (0)(-1)Q(1)+2(1)+3$
$=5$

$\therefore$ the remainder when $P(x)$ is divided by $x-1$ is 5.
{Remainder theorem}

46 $|x-3|\leqslant\dfrac{x}{2}$

$\therefore\ -\dfrac{x}{2}\leqslant x-3\leqslant\dfrac{x}{2}$
$\therefore\ -x\leqslant 2x-6\leqslant x$
$\therefore\ 0\leqslant 3x-6\leqslant 2x$

If $0\leqslant 3x-6$, then $3x\geqslant 6$ and so $x\geqslant 2$.
Also, $3x-6\leqslant 2x$, so $x-6\leqslant 0$ and so $x\leqslant 6$.
$\therefore\ 2\leqslant x\leqslant 6$

47 Let $P(x)=x^3+mx+m$

Now $P(m)=m$ {Remainder theorem}

$\therefore\ m^3+m^2+m=m$
$\therefore\ m^3+m^2=0$
$\therefore\ m^2(m+1)=0$
$\therefore\ m=0$ or -1

48 **a** Since $(x-1)^2$ is a factor of $P(x)$,

$$\begin{aligned}&x^4+ax^3+2x^2+bx-3\\&=(x-1)^2(x^2+cx+d) \text{ for some } c, d\\&=(x^2-2x+1)(x^2+cx+d)\\&=x^4+cx^3+dx^2-2x^3-2cx^2-2dx+x^2+cx+d\\&=x^4+(c-2)x^3+(d-2c+1)x^2+(c-2d)x+d\end{aligned}$$

Equating constant terms: $d=-3$

Equating coefficients of x^2: $-3-2c+1=2$

$\therefore\ c=-2$

Equating coefficients of x^3: $a=-2-2$

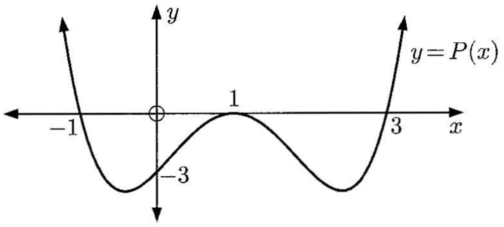
$\therefore\ a=-4$

Equating coefficients of x: $b=-2-2(-3)$

$\therefore\ b=4$

b So $P(x)=(x-1)^2(x^2-2x-3)$ {using c and d}

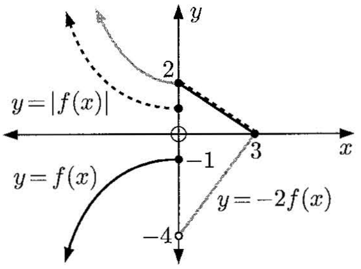
$=(x-1)^2(x-3)(x+1)$



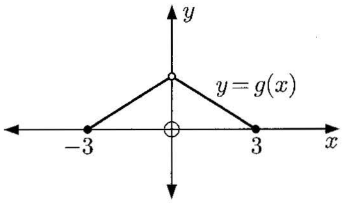
49 **a** Domain $=\{x \mid x\leqslant 3,\ x\in\mathbb{R}\}$

Range $=\{y \mid y\leqslant -1\cup 0\leqslant y<2,\ y\in\mathbb{R}\}$

b



c



i $g(x)$ is even.

ii $g(x)$ fails the horizontal line test, so $g(x)$ does not have an inverse.

50 Let $P(x)=x^3-x^2+(m+1)x+(2-m^2)$

Now $P(2)=0$

$\therefore\ 2^3-2^2+(m+1)2+(2-m^2)=0$

$\therefore\ 8-4+2m+2+2-m^2=0$

$\therefore\ m^2-2m-8=0$

$\therefore\ (m-4)(m+2)=0$

$\therefore\ m=-2$ or 4

If $m=-2$, $P(x)=x^3-x^2-x-2$

$=(x-2)(x^2+ax+1)$ for some a.

Equating coefficients of x^2 gives

$-1=-2+a$

$\therefore\ a=1$

$\therefore\ P(x)=(x-2)(x^2+x+1)$

x^2+x+1 has $\Delta=1^2-4\times1\times1=-3$, so no more real zeros exist.

If $m=4$, $P(x)=x^3-x^2+5x-14$

$=(x-2)(x^2+ax+7)$ for some a.

Equating coefficients of x^2 gives

$-1=-2+a$

$\therefore\ a=1$

$\therefore\ P(x)=(x-2)(x^2+x+7)$

x^2+x+7 has $\Delta=1^2-4\times1\times7=-27$, so no more real zeros exist.

51 $P(1)=6$ {Remainder theorem}

$\therefore\ 2+3+p=6$

$\therefore\ p=1$

$P(2)=77$ {Remainder theorem}

$\therefore\ 2^{m+1}+3\times2^n+1=77$

$\therefore\ 2^{m+1}=76-3\times2^n$

We need to find $n\in\mathbb{Z}^+$ such that $76-3\times2^n$ is a power of 2 which is $\geqslant 2^2$. $\{m\in\mathbb{Z}^+\}$

n	$76-3\times2^n$	
1	70	
2	64	✓
3	52	
4	28	
5	−20	too small

$\therefore\ n=2,\ 2^{m+1}=64$

$\therefore\ m=5$

So, the only solution is $m=5$, $n=2$, $p=1$.

52 $P(i)=0 \quad \therefore\ 6i^4+7i^3+8i^2+7i+k=0$

$\therefore\ 6-7i-8+7i+k=0$

$\therefore\ k=2$

Since $P(z)$ is a real polynomial, $-i$ must also be a zero.

$\therefore\ (z+i)(z-i)=z^2+1$ is a factor of $P(z)$

$\therefore\ P(z)=(z^2+1)(6z^2+az+2)$ for some a

Equating coefficients of x^3, $a=7$

$\therefore\ P(z)=(z^2+1)(6z^2+7z+2)$

$=(z^2+1)(3z+2)(2z+1)$

$\therefore$ the zeros of $P(z)$ are $\pm i$, $-\frac{2}{3}$, and $-\frac{1}{2}$

53 $a>b>0$

$\therefore\ \dfrac{a}{a}>\dfrac{b}{a}$ $\{a>0\}$

$\therefore\ \dfrac{a}{ab}>\dfrac{b}{ab}$ $\{b>0\}$

$\therefore\ \dfrac{1}{b}>\dfrac{1}{a}$

$\therefore\ \dfrac{1}{a}<\dfrac{1}{b}$

54 **a** $(f\circ g)(x)=2x-1$

$\therefore\ (g(x))^{\frac{1}{3}}=2x-1$

$\therefore\ g(x)=(2x-1)^3$

b $(g\circ f)(x)=2x-1$

$\therefore\ g(x^{\frac{1}{3}})=2x-1=2\left(x^{\frac{1}{3}}\right)^3-1$

$\therefore\ g(x)=2x^3-1$

55 $P(n) = 1000 + ae^{kn}$

Now $P(0) = 2000$

$\therefore\ 1000 + a = 2000$ and so $a = 1000$

Also, $P(12) = 4000$

$$\begin{aligned}\therefore\ 1000 + 1000e^{12k} &= 4000\\ \therefore\ 1000e^{12k} &= 3000\\ \therefore\ e^{12k} &= 3\\ \therefore\ e^{k} &= 3^{\frac{1}{12}}\end{aligned}$$

If $P(n) = 10\,000$, $1000 + ae^{kn} = 10\,000$

$$\begin{aligned}\therefore\ 1000e^{kn} &= 9000\\ \therefore\ e^{kn} &= 9\\ \therefore\ 3^{\frac{n}{12}} &= 3^2\\ \therefore\ n &= 24 \text{ months}\end{aligned}$$

It will take 2 years for the population to reach 10 000.

56 **a** Since a is a solution of the equation,

$$\begin{aligned}3a^3 - 11a^2 + 8a &= 12a\\ \therefore\ 3a^3 - 11a^2 - 4a &= 0\\ \therefore\ a(3a^2 - 11a - 4) &= 0\\ \therefore\ a(3a+1)(a-4) &= 0\\ \therefore\ a &= 0,\ -\tfrac{1}{3},\text{ or } 4\end{aligned}$$

b If $a = 0$, $3x^3 - 11x^2 + 8x = 0$

$$\begin{aligned}\therefore\ x(3x^2 - 11x + 8) &= 0\\ \therefore\ x(3x-8)(x-1) &= 0\\ \therefore\ x &= 0,\ \tfrac{8}{3},\text{ or } 1\end{aligned}$$

If $a = -\frac{1}{3}$, $3x^3 - 11x^2 + 8x = 12(-\frac{1}{3})$

$\therefore\ 3x^3 - 11x^2 + 8x + 4 = 0$

$x = a$ is a solution, and so $(3x+1)$ must be a factor.

$\therefore\ 3x^3 - 11x^2 + 8x + 4 = (3x+1)(x^2 + ax + 4)$ for some a

Equating coefficients of x^2 gives $-11 = 1 + 3a$

$\therefore\ a = -4$

$$\begin{aligned}\therefore\ (3x+1)(x^2 - 4x + 4) &= 0\\ \therefore\ (3x+1)(x-2)^2 &= 0\\ \therefore\ x &= -\tfrac{1}{3} \text{ or } 2\end{aligned}$$

If $a = 4$, $3x^3 - 11x^2 + 8x = 12(4)$

$\therefore\ 3x^3 - 11x^2 + 8x - 48 = 0$

$x = a$ is a solution, so $(x-4)$ is a factor.

$\therefore\ 3x^3 - 11x^2 + 8x - 48 = (x-4)(3x^2 + ax + 12)$ for some a

Equating coefficients of x^2 gives $-11 = a - 12$

$\therefore\ a = 1$

$\therefore\ (x-4)(3x^2 + x + 12) = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\times 3\times 12}}{2\times 3}$$

$$x = 4 \text{ or } \frac{-1 \pm i\sqrt{143}}{6}$$

57 $P(x)$ is a real polynomial, so $3 + 2i$ must also be a zero of $P(x)$.

$(3+2i) + (3-2i) = 6$ and

$$\begin{aligned}(3+2i)(3-2i) &= 9 + 4\\ &= 13\end{aligned}$$

So, $x^2 - 6x + 13$ is a factor of $P(x)$.

$$\begin{aligned}\therefore\ &2x^3 + mx^2 - (m+1)x + (3 - 4m)\\ &= (x^2 - 6x + 13)(2x + b) \quad \text{for some } b\\ &= 2x^3 + (b-12)x^2 + (26 - 6b)x + 13b\end{aligned}$$

Equating coefficients of x^2: $m = b - 12$ .... (1)

Equating constants: $3 - 4m = 13b$ .... (2)

Substituting (1) into (2): $13b = 3 - 4(b - 12)$

$$\begin{aligned}\therefore\ 13b &= 3 - 4b + 48\\ \therefore\ 17b &= 51\\ \therefore\ b &= 3\\ \therefore\ m &= 3 - 12 = -9\end{aligned}$$

$\therefore\ P(x) = (x^2 - 6x + 13)(2x + 3)$

$\therefore$ the zeros of $P(x)$ are $3 \pm 2i$ and $-\frac{3}{2}$

58 $a^2 \times a^3 + a^2 - a^4 \times a - 2 = 0$

$$\begin{aligned}\therefore\ a^2 - 2 &= 0\\ \therefore\ a &= \pm\sqrt{2}\end{aligned}$$

$\therefore\ P(z) = 2z^3 + z^2 - 4z - 2$

Since $P(z)$ is the same whether $a = \pm\sqrt{2}$, both $z = \sqrt{2}$ and $z = -\sqrt{2}$ must be zeros of $P(z)$.

Hence $(z - \sqrt{2})(z + \sqrt{2}) = (z^2 - 2)$ is a factor of $P(z)$.

$\therefore\ P(z) = (z^2 - 2)(2z + 1)$

$\therefore$ the zeros of $P(x)$ are $\pm\sqrt{2}$ and $-\frac{1}{2}$

59
$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(2x^3)\\ &= 2(2x^3) - 1\\ &= 4x^3 - 1\end{aligned}$$

So the function $(f\circ g)^{-1}$ is $x = 4y^3 - 1$

$$\begin{aligned}\therefore\ 4y^3 &= x + 1\\ \therefore\ y^3 &= \frac{x+1}{4}\\ \therefore\ y &= \left(\frac{x+1}{4}\right)^{\frac{1}{3}}\end{aligned}$$

So, $(f\circ g)^{-1} : x \mapsto \left(\dfrac{x+1}{4}\right)^{\frac{1}{3}}$.

60 Let $P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

Since bi is a zero of the real polynomial $P(x)$, so is $-bi$.

$\therefore\ x^2 + b^2$ is a factor of $P(x)$

$$\begin{aligned}\therefore\ P(x) &= x^4 + 2x^3 + 8x^2 + 6x + 15\\ &= (x^2 + b^2)(x^2 + cx + d) \quad \text{for some } c, d\\ &= x^4 + cx^3 + (b^2 + d)x^2 + b^2cx + b^2d\end{aligned}$$

Equating the coefficients of x^3: $c = 2$

Equating the coefficients of x: $2b^2 = 6$

$\therefore\ b = \pm\sqrt{3}$

Equating the coefficients of x^2: $3 + d = 8$

$\therefore\ d = 5$

$\therefore\ P(x) = (x^2 + 3)(x^2 + 2x + 5)$

Now $x^2 + 2x + 5 = 0$

$$\text{when } x = \frac{-2 \pm \sqrt{2^2 - 4\times 1\times 5}}{2\times 1} = -1 \pm 2i$$

$\therefore$ the zeros are $\pm\sqrt{3}i$, $-1 \pm 2i$

SOLUTIONS TO TOPIC 3 (CIRCULAR FUNCTIONS AND TRIGONOMETRY)

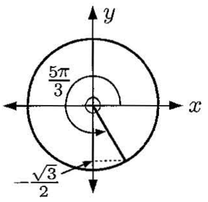
1 **a** $\frac{2\pi}{9}$ radians

$$\begin{aligned}&= \left(\tfrac{2\pi}{9} \times \tfrac{180}{\pi}\right)^\circ\\ &= 40^\circ\end{aligned}$$

b 140°

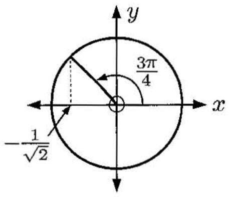
$$\begin{aligned}&= \left(140 \times \tfrac{\pi}{180}\right) \text{ radians}\\ &= \tfrac{7\pi}{9}\end{aligned}$$

2 **a**



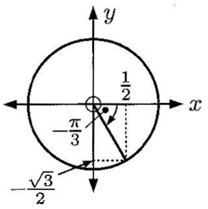
$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b



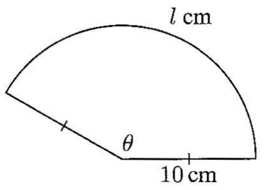
$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

c



$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

3



$$\begin{aligned} \text{perimeter} &= 40 \text{ cm} \\ \therefore\ 10 + 10 + l &= 40 \\ \therefore\ l &= 20 \\ \text{area} &= \tfrac{1}{2}\theta r^2 \\ &= \tfrac{1}{2}lr \quad \{l = \theta r\} \\ &= \tfrac{1}{2} \times 20 \times 10 \\ &= 100 \text{ cm}^2 \end{aligned}$$

4 **a** A vertical stretch with factor 2, and a horizontal stretch with factor 3.

b A translation of $\frac{\pi}{3}$ units to the left, and a translation of 4 units downwards.

5 **a** Amplitude = 1

The principal axis is $y = 0$

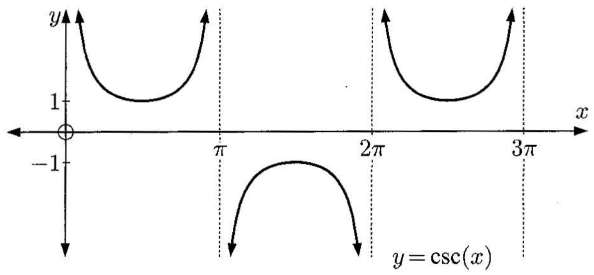
Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

b Amplitude = 2

The principal axis is $y = -1$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

6



7

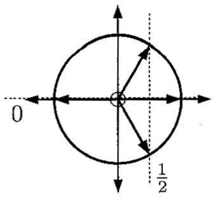
$$\begin{aligned} &\sin\left(\tfrac{3\pi}{2} - \phi\right)\tan(\phi + \pi) \\ &= \left(\sin\tfrac{3\pi}{2}\cos\phi - \cos\tfrac{3\pi}{2}\sin\phi\right)\tan\phi \\ &= ((-1)\cos\phi - 0 \times \sin\phi)\,\frac{\sin\phi}{\cos\phi} \\ &= -\sin\phi \end{aligned}$$

8

$$\begin{aligned} \cos 2x &= \tfrac{5}{8} \\ \therefore\ 1 - 2\sin^2 x &= \tfrac{5}{8} \quad \{\text{double angle formula}\} \\ \therefore\ 2\sin^2 x &= \tfrac{3}{8} \\ \therefore\ \sin x &= \pm\tfrac{\sqrt{3}}{4} \end{aligned}$$

9

$$\begin{aligned} \sin 2x &= \sin x, \quad x \in [-\pi, \pi] \\ \therefore\ 2\sin x\cos x - \sin x &= 0 \\ \therefore\ \sin x(2\cos x - 1) &= 0 \\ \therefore\ \sin x = 0 \ \text{ or } \ \cos x &= \tfrac{1}{2} \end{aligned}$$



$\therefore\ x = 0,\ \pm\frac{\pi}{3},\ \text{or}\ \pm\pi$

10

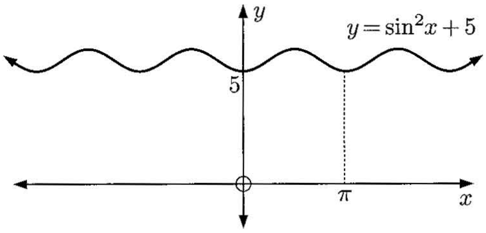
$$\begin{aligned} \text{area} &= 20 \text{ cm}^2 \\ \therefore\ \tfrac{1}{2}\theta r^2 &= 20 \\ \therefore\ \tfrac{1}{2}lr &= 20 \quad \{l = \theta r\} \\ \therefore\ \tfrac{1}{2}(6)r &= 20 \\ \therefore\ r &= \tfrac{20}{3} \text{ cm} \end{aligned}$$

$$\text{So,}\quad \theta = \frac{l}{r} = \frac{6}{\frac{20}{3}} = 0.9$$

11 **a** Period $= \frac{2\pi}{3}$

b Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

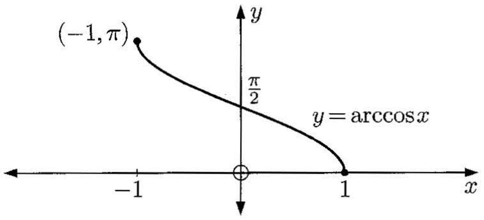
c



Period $= \pi$

$$\begin{aligned} \textit{or}\ \ \sin^2 x + 5 &= \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) + 5 \\ &= -\tfrac{1}{2}\cos 2x + \tfrac{11}{2} \\ \therefore\ \text{period} &= \tfrac{2\pi}{2} = \pi \end{aligned}$$

12



13

$$\begin{aligned} 1 - \frac{\sin^2\theta}{1 + \cos\theta} &= \frac{1 + \cos\theta}{1 + \cos\theta} - \frac{\sin^2\theta}{1 + \cos\theta} \\ &= \frac{1 + \cos\theta - (1 - \cos^2\theta)}{1 + \cos\theta} \\ &= \frac{\cos\theta + \cos^2\theta}{1 + \cos\theta} \\ &= \frac{\cos\theta(1 + \cos\theta)}{1 + \cos\theta} \\ &= \cos\theta \end{aligned}$$

14

$$\begin{aligned} \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2 \times 2}{1 - 2^2} \\ &= -\tfrac{4}{3} \end{aligned}$$

$$\begin{aligned} \tan 3\theta &= \tan(2\theta + \theta) \\ &= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} \\ &= \frac{-\frac{4}{3} + 2}{1 - \left(-\frac{4}{3}\right) \times 2} \\ &= \tfrac{2}{11} \end{aligned}$$

15 $\csc(2x) - \cot(2x) = \dfrac{1}{\sin 2x} - \dfrac{\cos 2x}{\sin 2x}$

$$= \frac{1-\cos 2x}{\sin 2x}$$

$$= \frac{1-(1-2\sin^2 x)}{2\sin x\cos x}$$

{double angle formulae}

$$= \frac{2\sin^2 x}{2\sin x\cos x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$\therefore\ \tan\left(\frac{5\pi}{12}\right) = \csc\left(\frac{5\pi}{6}\right) - \cot\left(\frac{5\pi}{6}\right)$

$$= \frac{1}{\sin\left(\frac{5\pi}{6}\right)} - \frac{1}{\tan\left(\frac{5\pi}{6}\right)}$$

$$= \frac{1}{\frac{1}{2}} - \frac{1}{-\frac{1}{\sqrt{3}}}$$

$$= 2+\sqrt{3}$$

16 $\cos 2\alpha = \sin^2\alpha$

$\therefore\ 1-2\sin^2\alpha = \sin^2\alpha$

$\therefore\ 1 = 3\sin^2\alpha$

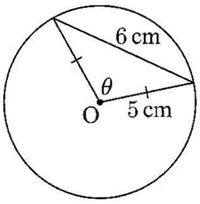
$\therefore\ \sin^2\alpha = \frac{1}{3}$

$\therefore\ \cos^2\alpha = \frac{2}{3}$

$\therefore\ \cot^2\alpha = \dfrac{\cos^2\alpha}{\sin^2\alpha} = 2$

$\therefore\ \cot\alpha = \pm\sqrt{2}$

17



$\cos\theta = \dfrac{5^2+5^2-6^2}{2\times5\times5}$ {cosine rule}

$\therefore\ \cos\theta = \dfrac{14}{50}$

$\therefore\ \theta \approx 1.287$

area of triangle $= \frac{1}{2}\times5\times5\times\sin\theta$

$= 12\ \text{cm}^2$

area of sector $= \frac{1}{2}\times\theta\times5^2$

$\approx 16.088\ \text{cm}^2$

$\therefore$ area of minor segment = area of sector − area of triangle

$\approx 4.09\ \text{cm}^2$

18 For the sine function $y = a\sin b(x-c)+d$:

The amplitude $= 2$, so $a = 2$.

The period $= \pi$, so $\frac{2\pi}{b} = \pi \Rightarrow b = 2$.

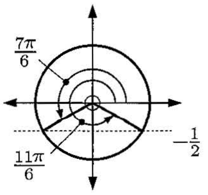
The principal axis is $y = 1$, so $d = 1$.

There is no horizontal translation, so $c = 0$.

$\therefore$ the function is $y = 2\sin(2x)+1$

We want to solve $2\sin(2x)+1 = 0,\ 0 \leqslant x \leqslant \pi$

$\therefore\ \sin 2x = -\frac{1}{2}$



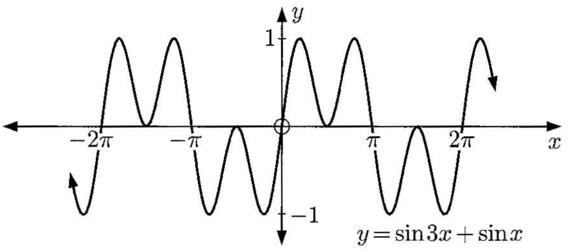
$\therefore\ 2x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$\therefore\ x = \frac{7\pi}{12}$ or $\frac{11\pi}{12}$

So, P is $\left(\frac{7\pi}{12}, 0\right)$ and Q is $\left(\frac{11\pi}{12}, 0\right)$.

19 **a** Period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$ **b** Period $= \frac{\pi}{5}$

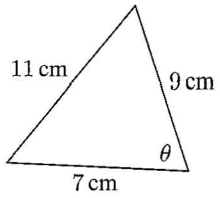
c



$y = \sin 3x$ has period $\frac{2\pi}{3}$, and $y = \sin x$ has period 2π.

So, $y = \sin 3x + \sin x$ has period 2π.

20



The largest angle is opposite the longest side.

$\cos\theta = \dfrac{9^2+7^2-11^2}{2\times7\times9}$ {cosine rule}

$\therefore\ \cos\theta = \frac{9}{126}$

$\therefore\ \theta \approx 85.9°$

21 **a** $\csc(x) = \dfrac{1}{\sin x}$

$\therefore$ vertical asymptotes occur when $\sin x = 0$

$\therefore$ the vertical asymptotes are $x = 0, \pm\pi$, and $\pm 2\pi$

b $\sec(2x) = \dfrac{1}{\cos 2x}$

$\therefore$ vertical asymptotes occur when $\cos 2x = 0$

$\therefore\ 2x = \pm\frac{\pi}{2} + k2\pi,\ k\in\mathbb{Z}$

$\therefore\ x = \pm\frac{\pi}{4} + k\pi,\ k\in\mathbb{Z}$

$\therefore$ the vertical asymptotes are

$x = \pm\frac{\pi}{4},\ \pm\frac{3\pi}{4},\ \pm\frac{5\pi}{4}$, and $\pm\frac{7\pi}{4}$.

c $\cot\left(\dfrac{x}{2}\right) = \dfrac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

$\therefore$ vertical asymptotes occur when $\sin\left(\dfrac{x}{2}\right) = 0$

$\therefore\ \dfrac{x}{2} = 0 + k\pi,\ k\in\mathbb{Z}$

$\therefore\ x = 2k\pi,\ k\in\mathbb{Z}$

$\therefore\ x = 0$ and $x = \pm2\pi$

$\therefore$ the vertical asymptotes are $x = 0$ and $x = \pm 2\pi$.

22 $\cos 79°\cos 71° - \sin 79°\sin 71° = \cos(79°+71°)$

$= \cos(150°)$

$= -\frac{\sqrt{3}}{2}$

23 $\tan 2A = \sin A$

$\therefore\ \dfrac{\sin 2A}{\cos 2A} = \sin A$

$\therefore\ \dfrac{2\sin A\cos A}{2\cos^2 A - 1} = \sin A$ {double angle formula}

$\therefore\ \dfrac{2\cos A}{2\cos^2 A - 1} = 1$ $\{\sin A \neq 0\}$

$2\cos A = 2\cos^2 A - 1$

$\therefore\ 2\cos^2 A - 2\cos A - 1 = 0$

$\therefore\ \cos A = \dfrac{2\pm\sqrt{(-2)^2-4\times2\times(-1)}}{2\times2}$

$= \dfrac{1\pm\sqrt{3}}{2}$

But $|\cos A| \leqslant 1$, so $\cos A = \dfrac{1-\sqrt{3}}{2}$

24 $\sin x - 2\cos x = A\sin(x+\alpha)$

$$\begin{aligned} &= A(\sin x\cos\alpha + \cos x \sin\alpha) \\ &= A\sin x\cos\alpha + A\cos x\sin\alpha \end{aligned}$$

Equating the coefficients of $\sin x$ and $\cos x$:

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = -2$$

$$\therefore \ \cos\alpha = \frac{1}{A} \quad \text{and} \quad \sin\alpha = \frac{-2}{A}$$

Now $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \ \left(\frac{-2}{A}\right)^2 + \left(\frac{1}{A}\right)^2 = 1$$

$$\therefore \ \frac{4+1}{A^2} = 1$$

$$\therefore \ A^2 = 5$$

$$\therefore \ A = \sqrt{5} \quad \{A > 0\}$$

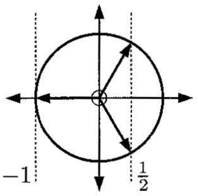
So $\cos\alpha = \frac{1}{\sqrt{5}}$, $\sin\alpha = -\frac{2}{\sqrt{5}}$

$\therefore$ α is in the 4th quadrant.

$\therefore$ $\alpha \approx 5.18$

25 $2\sin^2 x - \cos x = 1, \ x \in [0, 2\pi]$

$$\begin{aligned} \therefore \ & 2(1-\cos^2 x) - \cos x = 1 \\ \therefore \ & 2 - 2\cos^2 x - \cos x = 1 \\ \therefore \ & 2\cos^2 x + \cos x - 1 = 0 \\ \therefore \ & (2\cos x - 1)(\cos x + 1) = 0 \\ \therefore \ & \cos x = \tfrac{1}{2} \ \text{or} \ \cos x = -1 \end{aligned}$$

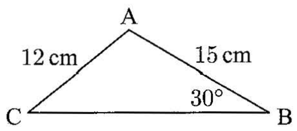


$\therefore \ x = \frac{\pi}{3}, \ \pi, \ \text{or} \ \frac{5\pi}{3}$

26 $\arcsin(2x-3) = -\frac{\pi}{6}$

$$\begin{aligned} \therefore \ & 2x - 3 = \sin\left(-\tfrac{\pi}{6}\right) \\ \therefore \ & 2x - 3 = -\tfrac{1}{2} \\ \therefore \ & 2x = \tfrac{5}{2} \\ \therefore \ & x = \tfrac{5}{4} \end{aligned}$$

27

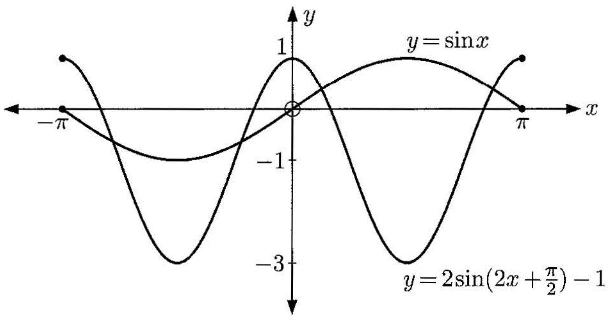


$$\frac{\sin C}{15} = \frac{\sin 30°}{12} \quad \{\text{sine rule}\}$$

$$\therefore \ C = \arcsin\left(\frac{15\sin 30°}{12}\right)$$

$$\therefore \ C \approx 38.7° \ \text{or} \ 141.3°$$

28



29 $\arcsin(-\frac{1}{2}) + \arctan(1) + \arccos(-\frac{1}{2})$

$$\begin{aligned} &= -\tfrac{\pi}{6} + \tfrac{\pi}{4} + \tfrac{2\pi}{3} \\ &= \tfrac{3\pi}{4} \end{aligned}$$

30 $\sin^2\theta + \cos^2\theta = 1$

$$\begin{aligned} \therefore \ & \left(\tfrac{2}{3}\right)^2 + \cos^2\theta = 1 \\ \therefore \ & \cos^2\theta = \tfrac{5}{9} \\ \therefore \ & \cos\theta = -\tfrac{\sqrt{5}}{3} \quad \{\theta \text{ is obtuse}\} \end{aligned}$$

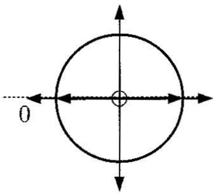
$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2 \times \tfrac{2}{3} \times -\tfrac{\sqrt{5}}{3} \\ &= -\tfrac{4\sqrt{5}}{9} \end{aligned}$$

31 $\sin x + \cos x = 1, \ 0 \leqslant x \leqslant \pi$

$\therefore \ \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 \quad .... \ (*)$

{squaring both sides}

$$\begin{aligned} \therefore \ & \sin 2x + 1 = 1 \\ \therefore \ & \sin 2x = 0 \end{aligned}$$



$$\begin{aligned} \therefore \ & 2x = 0 + k\pi, \ k \in \mathbb{Z} \\ \therefore \ & x = \tfrac{k\pi}{2}, \ k \in \mathbb{Z} \\ \therefore \ & x = 0, \ \tfrac{\pi}{2}, \ \pi \quad \{0 \leqslant x \leqslant \pi\} \end{aligned}$$

Since we squared both sides at $(*)$, we need to check our solutions.

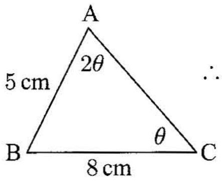
$\sin 0 + \cos 0 = 1$ ✓

$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$ ✓

$\sin\pi + \cos\pi = -1$ ✗

So $x = 0$ or $\frac{\pi}{2}$

32 **a**



$$\frac{\sin 2\theta}{8} = \frac{\sin\theta}{5} \quad \{\text{sine rule}\}$$

$$\therefore \ \frac{2\sin\theta\cos\theta}{8} = \frac{\sin\theta}{5}$$

$$\therefore \ \cos\theta = \tfrac{4}{5}$$

b

$$\begin{aligned} \widehat{ABC} &= \pi - 3\theta \\ &= \pi - 3\arccos(\tfrac{4}{5}) \\ &\approx 1.211 \end{aligned}$$

$$\begin{aligned} \text{Area of triangle} &\approx \tfrac{1}{2} \times 5 \times 8 \times \sin(1.211) \\ &\approx 18.7 \text{ cm}^2 \end{aligned}$$

33 $\tan 2x$ has period $\frac{\pi}{2}$, and $\tan 3x$ has period $\frac{\pi}{3}$.

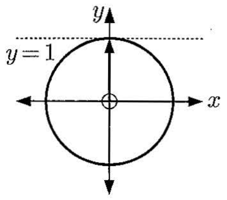
The lowest common multiple of $\frac{\pi}{2}$ and $\frac{\pi}{3}$ is π.

$\therefore$ the period $= \pi$

34 $\cot\theta + \tan\theta = 2, \ \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$$\therefore \ \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = 2$$

$$\therefore \ \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = 2$$

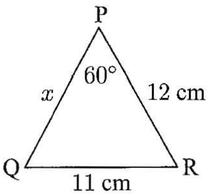


$$\begin{aligned} \therefore \ & 1 = 2\sin\theta\cos\theta \\ \therefore \ & 1 = \sin 2\theta \\ \therefore \ & 2\theta = \tfrac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z} \\ \therefore \ & \theta = \tfrac{\pi}{4} + k\pi, \ k \in \mathbb{Z} \\ \therefore \ & \theta = \tfrac{\pi}{4} \quad \{\text{since } \theta \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\} \end{aligned}$$

35
$$\tan 2\theta = 2$$
$$\therefore \quad \frac{2\tan\theta}{1-\tan^2\theta} = 2 \qquad \{\text{double angle formula}\}$$
$$2\tan\theta = 2 - 2\tan^2\theta$$
$$\tan^2\theta + \tan\theta - 1 = 0$$
$$\therefore \quad \tan\theta = \frac{-1 \pm \sqrt{1^2 - 4\times 1\times(-1)}}{2\times 1}$$
$$= \frac{-1\pm\sqrt{5}}{2}$$

Now $2\theta \in [\pi, \frac{3\pi}{2}]$, so $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$
$$\therefore \quad \tan\theta = \frac{-1-\sqrt{5}}{2} \qquad \{\tan\theta < 0\}$$

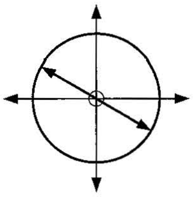
36



$$11^2 = x^2 + 12^2 - 2\times x\times 12\cos 60^\circ$$
$$\therefore \quad 121 = x^2 + 144 - 12x$$
$$\therefore \quad x^2 - 12x + 23 = 0$$
$$\therefore \quad x = \frac{12\pm\sqrt{(-12)^2 - 4\times 1\times 23}}{2\times 1}$$
$$\therefore \quad x = 6\pm\sqrt{13}, \text{ so } PQ = 6\pm\sqrt{13} \text{ cm}$$

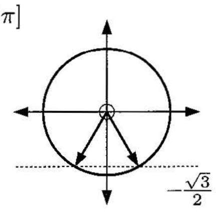
37
$$\frac{1}{\tan\theta - \sec\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}} \qquad \{\text{provided } \cos\theta \neq 0\}$$
$$= \frac{1}{\left(\frac{\sin\theta - 1}{\cos\theta}\right)}$$
$$= \frac{\cos\theta}{\sin\theta - 1} \times \left(\frac{\sin\theta+1}{\sin\theta+1}\right)$$
$$= \frac{\cos\theta\sin\theta + \cos\theta}{\sin^2\theta - 1}$$
$$= \frac{\cos\theta\sin\theta+\cos\theta}{-\cos^2\theta}$$
$$= -\frac{\cos\theta\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\cos^2\theta}$$
$$= -\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}$$
$$= -(\tan\theta + \sec\theta)$$

38 **a** $\sqrt{3}\tan\left(\frac{x}{2}\right) = -1, \ x \in [-\pi, 3\pi]$



$$\therefore \quad \tan\left(\frac{x}{2}\right) = -\tfrac{1}{\sqrt{3}}$$
$$\therefore \quad \frac{x}{2} = \tfrac{5\pi}{6} + k\pi, \ k\in\mathbb{Z}$$
$$\therefore \quad x = \tfrac{5\pi}{3} + 2k\pi, \ k\in\mathbb{Z}$$
$$\therefore \quad x = \tfrac{5\pi}{3} \text{ or } -\tfrac{\pi}{3} \quad \{x\in[-\pi, 3\pi]\}$$

b $\sqrt{3} + 2\sin(2x) = 0, \quad x \in [-\pi, 3\pi]$



$$\therefore \quad \sin 2x = -\tfrac{\sqrt{3}}{2}$$
$$\therefore \quad 2x = \left.\begin{matrix}\frac{4\pi}{3}\\ \frac{5\pi}{3}\end{matrix}\right\} + k2\pi, \ k\in\mathbb{Z}$$
$$\therefore \quad x = \left.\begin{matrix}\frac{2\pi}{3}\\ \frac{5\pi}{6}\end{matrix}\right\} + k\pi, \ k\in\mathbb{Z}$$
$$\therefore \quad x = -\tfrac{\pi}{3}, -\tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{5\pi}{6}, \tfrac{5\pi}{3}, \tfrac{11\pi}{6}, \tfrac{8\pi}{3}, \text{ or } \tfrac{17\pi}{6}$$
$$\{x\in[-\pi, 3\pi]\}$$

39
$$\sin x = 2\sin(x - \tfrac{\pi}{6})$$
$$\therefore \quad \sin x = 2(\sin x\cos\tfrac{\pi}{6} - \cos x\sin\tfrac{\pi}{6})$$
$$= 2\sin x(\tfrac{\sqrt{3}}{2}) - 2\cos x(\tfrac{1}{2})$$
$$\therefore \quad \sin x(1-\sqrt{3}) = -\cos x$$
$$\therefore \quad \frac{\sin x}{\cos x} = -\frac{1}{1-\sqrt{3}}$$
$$\therefore \quad \tan x = \frac{1}{\sqrt{3}-1}$$

40 **a** Consider the sine function model $H(t) = a\sin b(t-c)+d$

The amplitude $a = \frac{2.4}{2} = 1.2$ m.

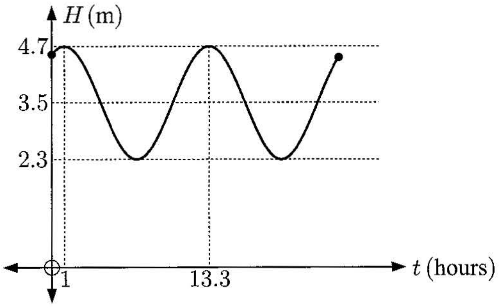
The period is 12.3 hours, so $\dfrac{2\pi}{b} = 12.3$
$$\therefore \quad b \approx 0.5108$$

The principal axis is $H = 4.7 - 1.2 = 3.5$ m so $d = 3.5$.

The first low tide is at $t = 1 + 6.15 = 7.15$, and the next high tide is at $t = 1 + 12.3 = 13.3$
$$\therefore \quad c = \frac{7.15+13.3}{2} \approx 10.2$$

$\therefore$ the model is $H(t) \approx 1.2\sin(0.5108(t-10.2)) + 3.5$ where t is the time in hours after midnight, $0 \leqslant t \leqslant 24$.

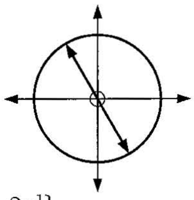
b



41 **a**
$$\sin\left(\arccos(-\tfrac{\sqrt{3}}{2})\right)$$
$$= \sin\left(\tfrac{5\pi}{6}\right)$$
$$= \tfrac{1}{2}$$

b
$$\tan\left(\arcsin\tfrac{1}{\sqrt{2}}\right)$$
$$= \tan\left(\tfrac{\pi}{4}\right)$$
$$= 1$$

42 $\sin x + \sqrt{3}\cos x = 0$

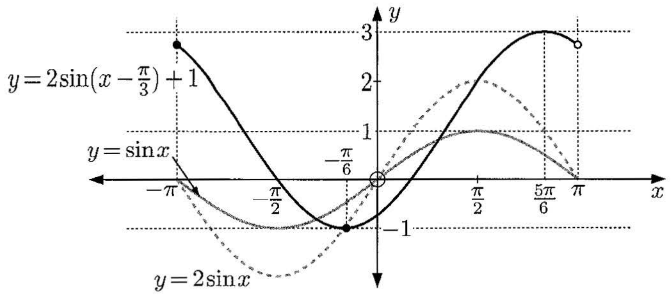


$$\therefore \quad \sin x = -\sqrt{3}\cos x$$
$$\therefore \quad \frac{\sin x}{\cos x} = -\sqrt{3}$$
$$\therefore \quad \tan x = -\sqrt{3}$$
$$\therefore \quad x = \tfrac{2\pi}{3} \text{ or } \tfrac{5\pi}{3} \quad \{x\in[0, 2\pi]\}$$

43
$$\frac{\sin\theta + 2\cos\theta}{\sin\theta - \cos\theta} = 2$$
$$\therefore \quad \sin\theta + 2\cos\theta = 2(\sin\theta - \cos\theta)$$
$$\therefore \quad 4\cos\theta = \sin\theta$$
$$\therefore \quad \tan\theta = 4$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\times 4}{1-4^2}$$
$$= -\tfrac{8}{15}$$

44 $y = 2\sin(x - \frac{\pi}{3}) + 1$ is a translation of $y = 2\sin x$ by $(\frac{\pi}{3}, 1)$.

So, we start with $y = \sin x$, we stretch it vertically with scale factor 2 to produce $y = 2\sin x$, then perform the translation.



45

$$\cos 2x+\sqrt{3}\sin 2x=1, \quad x\in[-\pi,\,\pi]$$

$$\therefore\ 1-2\sin^2 x+2\sqrt{3}\sin x\cos x=1$$

$$\therefore\ -2\sin^2 x+2\sqrt{3}\sin x\cos x=0$$

$$\therefore\ 2\sin x(\sqrt{3}\cos x-\sin x)=0$$

$$\therefore\ \sin x=0 \text{ or } \sqrt{3}\cos x=\sin x$$

$$\therefore\ \sin x=0 \text{ or } \tan x=\sqrt{3}$$

$$\therefore\ x=-\pi,\ -\tfrac{2\pi}{3},\ 0,\ \tfrac{\pi}{3},\text{ or }\pi$$

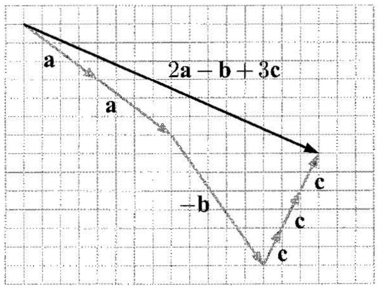
SOLUTIONS TO TOPIC 4 (VECTORS)

1 **a** $\mathbf{a}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

b magnitude $=\sqrt{1^2+(-2)^2+3^2}=\sqrt{14}$ units

c The unit vector in the opposite direction is $-\frac{1}{\sqrt{14}}\begin{pmatrix}1\\-2\\3\end{pmatrix}$.

2

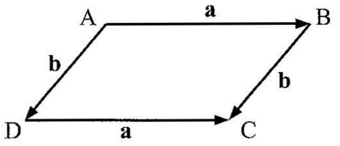


$$2\mathbf{a}-\mathbf{b}+3\mathbf{c}=2\begin{pmatrix}4\\-3\end{pmatrix}-\begin{pmatrix}-5\\7\end{pmatrix}+3\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}8+5+3\\-6-7+6\end{pmatrix}$$

$$=\begin{pmatrix}16\\-7\end{pmatrix}\ \text{which checks with the diagram.}$$

3 **a**



b $\overrightarrow{BC}=\overrightarrow{AD}=\mathbf{b}$ and $\overrightarrow{CD}=-\overrightarrow{AB}=-\mathbf{a}$

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{a}+\mathbf{b}$ and $\overrightarrow{BD}=\overrightarrow{BC}+\overrightarrow{CD}=\mathbf{b}-\mathbf{a}$

c

$$\overrightarrow{AC}\bullet\overrightarrow{BD}=(\mathbf{a}+\mathbf{b})\bullet(\mathbf{b}-\mathbf{a})$$

$$=\mathbf{a}\bullet\mathbf{b}-\mathbf{a}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}-\mathbf{b}\bullet\mathbf{a}$$

$$=\mathbf{b}\bullet\mathbf{b}-\mathbf{a}\bullet\mathbf{a}\quad\{\text{as }\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}\}$$

$$=|\mathbf{b}|^2-|\mathbf{a}|^2$$

and, if $|\mathbf{b}|=|\mathbf{a}|$, $\overrightarrow{AC}\bullet\overrightarrow{BD}=0$

d Since $\overrightarrow{AC}\bullet\overrightarrow{BD}=0$, $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular.

4 **a** $\mathbf{a}\bullet\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
If $\mathbf{a}\bullet\mathbf{b}<0$, then $\cos\theta<0$ and so $90°<\theta<180°$.

b

$$\mathbf{a}\bullet\mathbf{b}=\begin{pmatrix}-2\\1\\3\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\1\end{pmatrix}=-6-1+3=-4$$

$|\mathbf{a}|=\sqrt{(-2)^2+1^2+3^2}=\sqrt{14}$ and

$|\mathbf{b}|=\sqrt{3^2+(-1)^2+1^2}=\sqrt{11}$

and $\cos\theta=\dfrac{-4}{\sqrt{14}\sqrt{11}}\approx-0.3223$ and so $\theta\approx108.8°$.

5 **a** $\begin{pmatrix}k\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\k\\3k\end{pmatrix}$ are parallel if $\begin{pmatrix}4\\k\\3k\end{pmatrix}=a\begin{pmatrix}k\\1\\3\end{pmatrix}$

for some a.

Thus, $4=ak$ .... (1)
$k=a$ .... (2)
$3k=3a$ .... (3)

From (1) and (2), $k^2=4$ and so $k=\pm2$.
Hence the vectors are parallel if $k=\pm2$.

- if $k=2$, the vectors are $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\2\\6\end{pmatrix}$
- if $k=-2$, the vectors are $\begin{pmatrix}-2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\-2\\-6\end{pmatrix}$

b The vectors are perpendicular if $\begin{pmatrix}k\\1\\3\end{pmatrix}\bullet\begin{pmatrix}4\\k\\3k\end{pmatrix}=0$

$\therefore\ 4k+k+9k=0$ and so $k=0$.

So, the vectors $\begin{pmatrix}0\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\0\\0\end{pmatrix}$ are perpendicular.

6 **a** **i** The equation can be written as

$$\frac{x-1}{2}=\frac{3-y}{3}=z=t$$

$$\therefore\ x=2t+1,\ y=-3t+3,\ z=t$$

$$\text{or}\quad\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\0\end{pmatrix}+t\begin{pmatrix}2\\-3\\1\end{pmatrix}.$$

$\therefore$ a vector parallel to the line is $\begin{pmatrix}2\\-3\\1\end{pmatrix}$.

ii Letting $t=0$, a point on the line is $(1,3,0)$.

iii The point $(7,-3,2)$ lies on the line if

$7=2t+1$ .... (1)
$-3=-3t+3$ .... (2)
$2=t$ .... (3)

So, from (3), $t=2$ and from (1), $t=3$ which is not possible. Thus the point $(7,-3,2)$ does not lie on the line.

b There are many possible answers.

Since $\begin{pmatrix}0\\1\\3\end{pmatrix}$ is perpendicular to $\begin{pmatrix}2\\-3\\1\end{pmatrix}$, a possible line is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\-3\\2\end{pmatrix}+s\begin{pmatrix}0\\1\\3\end{pmatrix}$, which is $x=5$, $y=-3+s$, $z=2+3s$, $s\in\mathbb{R}$.

c If the lines in **a** and **b** are to meet, then

$$\begin{aligned} 5 &= 2t+1 && \text{.... (1)} \\ -3+s &= -3t+3 && \text{.... (2)} \\ 2+3s &= t && \text{.... (3)} \end{aligned}$$

and from these we see that $t = 2$ and $s = 0$

So, the lines meet at the point $(5, -3, 2)$.

d The line in **a** has direction vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Line L has direction vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

If the angle between the lines is θ, then

$$\left| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right| = \sqrt{4+9+1}\sqrt{1+4+1}\cos\theta$$

$$\therefore\ |-2-6+1| = \sqrt{14}\sqrt{6}\cos\theta$$

$$\therefore\ \cos\theta = \frac{7}{\sqrt{84}} \text{ and so } \theta \approx 40.2^\circ$$

7 If two lines have two points in common, they coincide. Alternatively, if two lines have the same direction vector and one point in common they coincide.

8 An equation of a plane through $(3, -1, 2)$ parallel to the vectors $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

$$\begin{aligned} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 3 \\ 3 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}\mathbf{k} \\ &= (1 - -3)\mathbf{i} - (-2-9)\mathbf{j} + (2-3)\mathbf{k} \\ &= 4\mathbf{i} + 11\mathbf{j} - \mathbf{k} \end{aligned}$$

So, the equation of the plane is

$$\begin{aligned} &4x + 11y - z = 4(3) + 11(-1) - 2 \\ \therefore\ &4x + 11y - z = -1 \end{aligned}$$

9 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 - -1 \\ 1-2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$,

$\overrightarrow{AC} = \begin{pmatrix} 4 - -1 \\ -3-2 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$

One equation for the plane is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$$

b A normal to the plane in **a** is

$$\begin{aligned} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 5 & -5 & 4 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 2 \\ -5 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 5 & -5 \end{vmatrix}\mathbf{k} \\ &= (-4+10)\mathbf{i} - (12-10)\mathbf{j} + (-15+5)\mathbf{k} \\ &= 6\mathbf{i} - 2\mathbf{j} - 10\mathbf{k} \\ &= 2(3\mathbf{i} - \mathbf{j} - 5\mathbf{k}) \end{aligned}$$

So, $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$, and the line has direction vector $\mathbf{l} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

If ϕ is the acute angle between the normal and the line,

$$\begin{aligned} \phi &= \sin^{-1}\left(\frac{|3+1-5|}{\sqrt{9+1+25}\sqrt{1+1+1}}\right) \\ &= \sin^{-1}\left(\frac{1}{\sqrt{35}\sqrt{3}}\right) \\ &\approx 5.60^\circ \end{aligned}$$

10 **a** A vector normal to $3x - 2y + 7z = 6$ is $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$.

b There are many possible solutions. Letting $y = z = 0$ gives $(2, 0, 0)$ as a point on the plane.

c A line through $(2, -1, 1)$ and normal to the plane has equation $\mathbf{l} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$.

This line meets the plane when

$$\begin{aligned} &3(2+3\lambda) - 2(-1-2\lambda) + 7(1+7\lambda) = 6 \\ \therefore\ &6 + 9\lambda + 2 + 4\lambda + 7 + 49\lambda = 6 \\ \therefore\ &\lambda = -\tfrac{9}{62} \end{aligned}$$

$\therefore$ the intersection of the line and plane is at $(2 + 3(-\frac{9}{62}), -1 - 2(-\frac{9}{62}), 1 + 7(-\frac{9}{62}))$ or $(\frac{97}{62}, -\frac{22}{31}, -\frac{1}{62})$.

$\therefore$ the shortest distance from $(2, -1, 1)$ to the plane

$$\begin{aligned} &= \sqrt{(2 - \tfrac{97}{62})^2 + (-1 - (-\tfrac{22}{31}))^2 + (1 - (-\tfrac{1}{62}))^2} \\ &= \sqrt{\frac{5022}{3844}} = \frac{9}{\sqrt{62}} \text{ units.} \end{aligned}$$

11 **a** The planes $2x + 4y + z = 1$ and $3x + 5y = 1$ have normals $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ respectively.

Let θ be the angle between the normals. Then

$$\left| \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right| = \sqrt{4+16+1}\sqrt{9+25+0}\cos\theta$$

$$\therefore\ |6+20+0| = \sqrt{21}\sqrt{34}\cos\theta$$

$$\therefore\ \cos\theta = \frac{26}{\sqrt{21}\sqrt{34}} \text{ and so } \theta \approx 13.3^\circ$$

b
$$\begin{aligned} 2x + 4y + z &= 1 && \text{.... (1)} \\ 3x + 5y &= 1 && \text{.... (2)} \end{aligned}$$

Let $x = t$, then in (2), $y = \dfrac{1-3t}{5}$

and so in (1), $2t + 4\left(\dfrac{1-3t}{5}\right) + z = 1$

$$\begin{aligned} \therefore\ &10t + 4 - 12t + 5z = 5 \\ \therefore\ &5z = 1 + 2t \\ \therefore\ &z = \frac{1+2t}{5} \end{aligned}$$

So, the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} + t\begin{pmatrix} 1 \\ -\frac{3}{5} \\ \frac{2}{5} \end{pmatrix}$, $t \in \mathbb{R}$

This solution is an equation of the line of intersection of the two planes.

c If the points lie on the plane $5x + 13y + 7z = 4$ then

$$5t + 13\left(\frac{1-3t}{5}\right) + 7\left(\frac{1+2t}{5}\right) = 4$$

$$\therefore\ 25t + 13 - 39t + 7 + 14t = 20$$

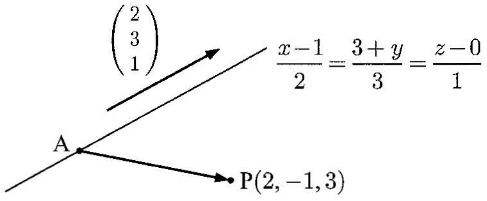
$$\therefore\ 20 = 20$$

Hence, the line of intersection of the first two planes lies on the third plane. This means that the infinite number of points on the line are the solutions of all three equations.

12 The equation of the line in parametric form is $x = 1 + 2t,\ y = -3 + 3t,\ z = t.$

$\therefore$ any point A on this line has coordinates $(1 + 2t,\ -3 + 3t,\ t)$.

$\therefore$ $\overrightarrow{PA} = \begin{pmatrix} -1+2t \\ -2+3t \\ -3+t \end{pmatrix}$ and the line has direction vector $\mathbf{l} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.



The shortest distance occurs when $\overrightarrow{PA} \bullet \mathbf{l} = 0$

$$\therefore\ -2 + 4t - 6 + 9t - 3 + t = 0$$

$$\therefore\ 14t = 11 \quad \therefore\ t = \tfrac{11}{14}$$

Thus $\overrightarrow{PA} = \frac{1}{14}\begin{pmatrix} 8 \\ 5 \\ -31 \end{pmatrix}$

and the distance $= \frac{1}{14}\sqrt{8^2 + 5^2 + (-31)^2} \approx 2.31$ units.

13 **a** $\overrightarrow{AB} = \begin{pmatrix} 5-1 \\ -1--1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

So, L has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix},\ t \in \mathbb{R}.$

b The plane has normal vector $\mathbf{n} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ and passes through $A(1, -1, 2)$

$\therefore$ the equation of the plane is

$$4x + 0y - 3z = 4(1) + 0(-1) - 3(2)$$

$$\therefore\ 4x - 3z = -2$$

c Any point P on L has coordinates $(1 + 4t,\ -1,\ 2 - 3t)$.

$$\therefore\ \overrightarrow{AP} = \begin{pmatrix} 4t \\ 0 \\ -3t \end{pmatrix}$$

Now $|\overrightarrow{AP}| = 20$, so $\sqrt{(4t)^2 + (-3t)^2} = 20$

$$\therefore\ 16t^2 + 9t^2 = 400$$

$$\therefore\ t^2 = 16$$

$$\therefore\ t = \pm 4$$

So, letting $t = 4$, a point on L which is 20 units from A is $(1 + 4(4),\ -1,\ 2 - 3(4)) = (17,\ -1,\ -10)$.

14 **a** When $t = 0$, $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

$\therefore$ ship A is at $(-1, 3)$ and ship B is at $(7, 4)$.

b Ship A has velocity vector $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$\therefore$ its speed is $\sqrt{4^2 + (-1)^2} = \sqrt{17}\ \text{km h}^{-1}$.

Ship B has velocity vector $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\therefore$ its speed is $\sqrt{(-2)^2 + (-1)^2} = \sqrt{5}\ \text{km h}^{-1}$.

c Suppose the ships pass through the same point, and that ship A is there at time t_A and ship B is there at time t_B.

$$x_A = x_B \ \Rightarrow\ -1 + 4t_A = 7 - 2t_B$$

$$\therefore\ 4t_A + 2t_B = 8$$

$$\therefore\ 2t_A + t_B = 4 \quad .... (1)$$

$$y_A = y_B \ \Rightarrow\ 3 - t_A = 4 - t_B$$

$$\therefore\ t_A - t_B = -1 \quad .... (2)$$

$(1) + (2)$ gives $3t_A = 3$

$$\therefore\ t_A = 1$$

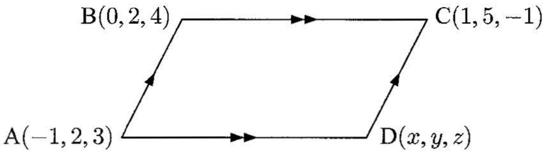
Using (2), $t_B = 2$

When $t_A = 1$, $x_A = -1 + 4 = 3$

and $y_A = 3 - 1 = 2$

So, the two ships both pass through $(3, 2)$. Ship A is there after 1 hour, and ship B is there after 2 hours.

15



a $\overrightarrow{AD} = \overrightarrow{BC}$

$$\therefore\ \begin{pmatrix} x+1 \\ y-2 \\ z-3 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 5-2 \\ -1-4 \end{pmatrix}$$

$\therefore\ x + 1 = 1,\ y - 2 = 3,\ z - 3 = -5$

$\therefore\ x = 0,\ y = 5,\ z = -2$

So, D is $(0, 5, -2)$.

b $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 3 & -5 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 0 & 1 \\ 3 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}$$

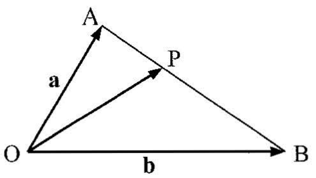
$$= -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$$

$$\therefore\ \text{area} = |-3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}|$$

$$= \sqrt{(-3)^2 + 6^2 + 3^2}$$

$$= 3\sqrt{6}\ \text{units}^2$$

16 **a**



$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

$$\therefore\ \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$= \overrightarrow{OA} + t\overrightarrow{AB}$$

$$= \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

$$= (1 - t)\mathbf{a} + t\mathbf{b}$$

b $\overrightarrow{AP} : \overrightarrow{PB} = 2 : 5 \quad \therefore\ \overrightarrow{AP} = \frac{2}{7}\overrightarrow{AB}$

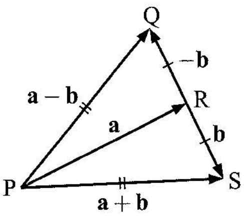
$$\therefore\ \mathbf{p} = \tfrac{5}{7}\mathbf{a} + \tfrac{2}{7}\mathbf{b}$$

$$= \tfrac{5}{7}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \tfrac{2}{7}\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{7} \\ -\frac{3}{7} \\ \frac{18}{7} \end{pmatrix}$$

So, P is $\left(\frac{4}{7},\ -\frac{3}{7},\ \frac{18}{7}\right)$.

17



If $|\mathbf{a}-\mathbf{b}| = |\mathbf{a}+\mathbf{b}|$, then PQ = PS, and triangle PQS is isosceles.

Since $|-\mathbf{b}| = |\mathbf{b}|$, QR = SR

$\therefore$ $\overrightarrow{PR} \perp \overrightarrow{QS}$ {the line joining the apex of an isosceles triangle to the midpoint of the base is perpendicular to the base}

So, **a** is perpendicular to **b**.

18 **a** $\overrightarrow{AB} = \begin{pmatrix} 0 - -1 \\ 1-2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ line (AB) has equation

$$\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

b The line L has direction vector $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

and (AB) has direction vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

If θ is the angle between the lines, then

$$\cos\theta = \frac{2+0-2}{\sqrt{2^2+(-1)^2}\sqrt{1^2+(-1)^2+2^2}} = 0$$

$$\therefore \ \theta = 90^\circ$$

So, the angle between (AB) and L is 90°.

19 $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{c} - 2\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 3-2k \end{pmatrix}$

Now **b** is perpendicular to $\mathbf{c} - 2\mathbf{a}$

$$\therefore \ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -4 \\ 3-2k \end{pmatrix} = 0$$

$$\therefore \ -6 - 3 + 2k = 0$$

$$\therefore \ 2k = 9$$

$$\therefore \ k = \tfrac{9}{2}$$

20 The line has parametric equations $x = 1$, $y = 2 - \lambda$, $z = 1 + 2\lambda$, $\lambda \in \mathbb{R}$.

The plane has parametric equations $x = 3 + s - t$, $y = s + 2t$, $z = -1 + s + 4t$, $s, t \in \mathbb{R}$.

So, the line and the plane meet when

$1 = 3 + s - t$ $\quad \therefore \ t = s + 2$ .... (1)

$2 - \lambda = s + 2t$ $\quad \therefore \ \lambda = 2 - s - 2t$ .... (2)

$1 + 2\lambda = -1 + s + 4t$ .... (3)

Substituting (1) into (2) gives

$$\lambda = 2 - s - 2(s+2)$$

$\therefore \ \lambda = -3s - 2$ .... (4)

Substituting (1) and (4) into (3) gives

$$1 + 2(-3s-2) = -1 + s + 4(s+2)$$

$$\therefore \ -6s - 3 = 5s + 7$$

$$\therefore \ 11s = -10$$

$$\therefore \ s = -\tfrac{10}{11}$$

$$\therefore \ \lambda = -3\left(-\tfrac{10}{11}\right) - 2 = \tfrac{8}{11}$$

$\therefore$ the point of intersection is $\left(1, 2 - \frac{8}{11}, 1 + 2\left(\frac{8}{11}\right)\right)$ which is $\left(1, \frac{14}{11}, \frac{27}{11}\right)$.

21 **a** P_1 has normal vector $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and

P_2 has normal vector $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

If θ is the acute angle between the planes, then

$$\cos\theta = \frac{|2+0-1|}{\sqrt{2^2+1^2+1^2}\sqrt{1^2+(-1)^2}}$$

$$\therefore \ \cos\theta = \frac{1}{\sqrt{6}\sqrt{2}}$$

$$\therefore \ \theta \approx 73^\circ$$

b The Cartesian equation of P_1 is $2x + y + z = 2$, and the Cartesian equation of P_2 is $x - z = 5$.

The planes are clearly not parallel.

Letting $z = t$, $x = 5 + t$

and so $2(5+t) + y + t = 2$

$\therefore \ y = -8 - 3t$

So, the equation of the line of intersection is

$$\mathbf{r} = \begin{pmatrix} 5 \\ -8 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}.$$

22 The line has parametric equations $x = 1 + \lambda$, $y = 1 + 2\lambda$, $z = -\lambda$.

This line meets the plane $3x + 2y - z = 1$ where

$$3(1+\lambda) + 2(1+2\lambda) - (-\lambda) = 1$$

$$\therefore \ 3 + 3\lambda + 2 + 4\lambda + \lambda = 1$$

$$\therefore \ 8\lambda = -4$$

$$\therefore \ \lambda = -\tfrac{1}{2}$$

So, P is $\left(1 - \frac{1}{2}, \ 1 + 2\left(-\frac{1}{2}\right), \ -\left(-\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$

23 $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 4 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 0 & 4 \end{vmatrix}\mathbf{k}$$

$$= -7\mathbf{i} - \mathbf{j} - 4\mathbf{k}$$

$$\therefore \ (\mathbf{b} \times \mathbf{c}) \bullet 2\mathbf{a} = \begin{pmatrix} -7 \\ -1 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = -14 - 2 - 16 = -32$$

24 **a** Equating x, y, and z values for the two lines gives

$4 + t = -1 + 3s$ .... (1)

$3 + 2t = 1 - 2s$ .... (2)

$-1 - 2t = 2 + s$ .... (3)

Adding (2) and (3) gives

$$2 = 3 - s$$

$$\therefore \ s = 1$$

Using (2) with $s = 1$,

$$3 + 2t = 1 - 2$$
$$\therefore \ 2t = -4$$
$$\therefore \ t = -2$$

Substituting $s = 1$ and $t = -2$ into (1) gives $4 + (-2) = -1 + 3(1)$ which is consistent.

$\therefore$ the lines intersect at the point corresponding to $s = 1$ and $t = -2$, and this is the point $(2, -1, 3)$.

b Direction vectors for lines L_1 and L_2 are

$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ respectively.

If θ is the acute angle between **a** and **b**, then

$$\begin{aligned} \cos\theta &= \frac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} \\ &= \frac{|3 - 4 - 2|}{\sqrt{1+4+4}\sqrt{9+4+1}} \\ &= \frac{3}{3\sqrt{14}} \\ &= \frac{1}{\sqrt{14}} \end{aligned}$$

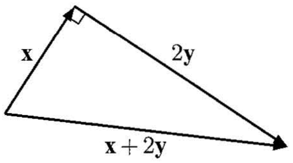
$$\therefore \ \theta \approx 74.5^\circ$$

$\therefore$ the acute angle between L_1 and L_2 is about 74.5°.

25 **a** If $\mathbf{y} = -2\mathbf{x}$

then $|\mathbf{x} + 2\mathbf{y}| = |\mathbf{x} - 4\mathbf{x}|$
$= |-3\mathbf{x}| = 3\,|\mathbf{x}| = 6$

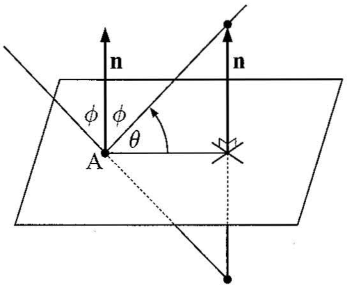
b



Since **x** and **y** are perpendicular, **x** and 2**y** are perpendicular. Using Pythagoras' theorem,

$$\begin{aligned} |\mathbf{x} + 2\mathbf{y}| &= \sqrt{|\mathbf{x}|^2 + |2\mathbf{y}|^2} \\ &= \sqrt{|\mathbf{x}|^2 + 4\,|\mathbf{y}|^2} \\ &= \sqrt{|\mathbf{x}|^2 + 4(3\,|\mathbf{x}|)^2} \\ &= \sqrt{37\,|\mathbf{x}|^2} \\ &= 2\sqrt{37} \end{aligned}$$

26



The plane has 2 direction vectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

Hence, $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{vmatrix}$

$$\begin{aligned} &= \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}\mathbf{k} \\ &= (-2+1)\mathbf{i} - (2-0)\mathbf{j} + (-1-0)\mathbf{k} \\ &= -\mathbf{i} - 2\mathbf{j} - \mathbf{k} \\ &= -\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{is a normal to the plane.} \end{aligned}$$

The line has direction vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

An angle ϕ between the normal and the line is given by (see diagram)

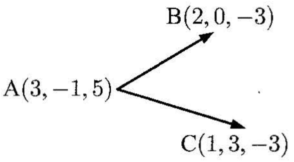
$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \sqrt{1+4+1}\sqrt{1+4+4}\cos\phi$$

$$\therefore \ 7 = \sqrt{6}\sqrt{9}\cos\phi$$

$$\therefore \ \cos\phi = \frac{7}{\sqrt{54}} \quad \text{and} \quad \phi \approx 0.309^c$$

Hence the angle between the line and its reflection is $2 \times \phi \approx 0.618^c$.

27



$$\overrightarrow{AB} = \begin{pmatrix} 2-3 \\ 0 - -1 \\ -3-5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -8 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 1-3 \\ 3 - -1 \\ -3-5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -8 \end{pmatrix}$$

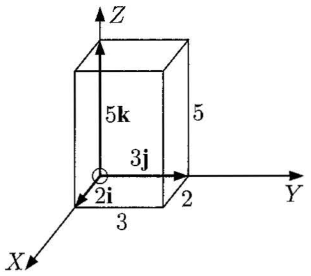
$$\overrightarrow{AB} \bullet \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos B\widehat{A}C$$

$$\therefore \ 2 + 4 + 64 = \sqrt{1+1+64}\sqrt{4+16+64}\cos B\widehat{A}C$$

$$\therefore \ \cos B\widehat{A}C = \frac{70}{\sqrt{66}\sqrt{84}}$$

$$\therefore \ B\widehat{A}C \approx 19.9^\circ$$

28



Diagonal from origin to $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ has direction $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

Diagonal from $5\mathbf{k}$ to $2\mathbf{i} + 3\mathbf{j}$ has direction $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$.

If the angle between the diagonals is θ then

$$\begin{aligned} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} &= \left|\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}\right| \cos\theta \\ &= \sqrt{38}\sqrt{38}\cos\theta \end{aligned}$$

$$\therefore \ 4 + 9 - 25 = 38\cos\theta$$

$$\therefore \ \cos\theta = -\tfrac{12}{38} \quad \text{and so} \quad \theta \approx 108.4^\circ$$

The acute angle between the diagonals is 71.6°.

Note: This angle may depend on the diagonals you select; the two other possible angles are 37.9° and 58.2°.

SOLUTIONS TO TOPIC 5 (STATISTICS AND PROBABILITY)

1 **a** Each horse is different, so the horses in the race will not have the same chance of winning the race. It is therefore impossible that a given horse will have exactly a $\frac{1}{10}$ chance of winning.

b Unless the outcomes are all equally likely,

$P(A) = \dfrac{n(A)}{n(U)}$ will not hold.

2 **a** There are 13 balls in the bucket, so the total number of ways of selecting 5 balls is $\binom{13}{5}$.

2 reds can be selected in $\binom{7}{2}$ ways.

1 blue can be selected in $\binom{3}{1}$ ways.

2 blacks can be selected in $\binom{2}{2}$ ways.

The probability of selecting 2 reds, 1 blue, and 2 blacks is

$$\frac{\binom{7}{2}\binom{3}{1}\binom{2}{2}}{\binom{13}{5}} \approx 0.0490$$

b Let X be the number of red balls selected.

$$\begin{aligned} \therefore\ P(X \geqslant 1) &= 1 - P(X = 0) \\ &= 1 - \frac{\binom{7}{0}\binom{6}{5}}{\binom{13}{5}} \approx 0.995 \end{aligned}$$

3 **a**
$$\begin{aligned} \frac{8+5+5+6+10+7+x+7+8}{9} &= 7 \\ \therefore\ x + 56 &= 63 \\ \therefore\ x &= 7 \end{aligned}$$

b In order, the scores are 5, 5, 6, 7, 7, 7, 8, 8, 10

$\therefore$ the median is 7.

c 7 is the most frequently occurring score, so the mode is 7.

d Since the mode, median, and mean are all 7, we expect the distribution to be approximately symmetrical.

4 **a** $f(x) \geqslant 0$ on a given interval $[a, b]$, and $\int_a^b f(x)\,dx = 1$.

b $P(x_1 \leqslant X \leqslant x_2) = \int_{x_1}^{x_2} f(x)\,dx$

5 **a** **i** Events A and B are mutually exclusive if $A \cap B = \varnothing$. In this case $A \cap B$ contains the numbers 41, 42, 43, 44.

So, the events are not mutually exclusive.

ii
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \tfrac{44}{100} + \tfrac{14}{100} - \tfrac{4}{100} \\ &= \tfrac{54}{100} \end{aligned}$$

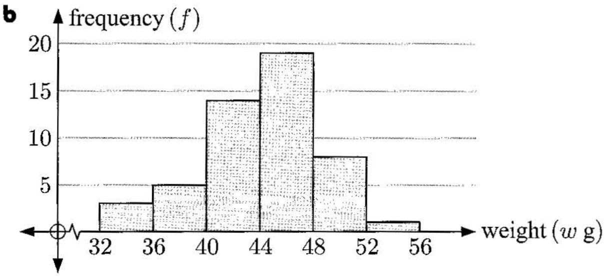
b If the events A and B are independent, $P(A \mid B) = P(A)$.

If A and B are mutually exclusive, $P(A \mid B) = 0$ since if B occurs A cannot occur.

But $P(A) \neq 0$. Hence, the events cannot be both independent and mutually exclusive.

6 **a** 50 chicks were weighed.

b



c

Midpoint weight (x g)	Frequency f	fx	$x-\mu$	$f(x-\mu)^2$
34	3	102	−10.16	309.68
38	5	190	−6.16	189.73
42	14	588	−2.16	65.32
46	19	874	1.84	64.33
50	8	400	5.84	272.84
54	1	54	9.84	96.83
Total	50	2208		998.72

mean $\mu = \dfrac{\sum fx}{n} = \dfrac{2208}{50} = 44.16$

$$\begin{aligned} \text{standard deviation}\quad \sigma &= \sqrt{\frac{\sum f(x-\mu)^2}{n}} \\ &= \sqrt{\frac{998.72}{50}} \\ &\approx 4.47 \end{aligned}$$

The mean weight was about 44.16 g with standard deviation 4.47 g.

7 **a** For $f(x)$ to be a probability density function,

$$\begin{aligned} \int_0^2 a(x^2+2)\,dx &= 1 \\ \therefore\ a\int_0^2 (x^2+2)\,dx &= 1 \\ \therefore\ a\left[\frac{x^3}{3} + 2x\right]_0^2 &= 1 \\ \therefore\ a\left(\tfrac{8}{3} + 4\right) &= 1 \\ \therefore\ a &= \tfrac{3}{20} \end{aligned}$$

b **i**
$$\begin{aligned} P(0.5 \leqslant X \leqslant 1.4) &= \int_{0.5}^{1.4} \tfrac{3}{20}(x^2+2)dx \\ &= \tfrac{3}{20}\left[\frac{x^3}{3} + 2x\right]_{0.5}^{1.4} \\ &= \tfrac{3}{20}\left(\tfrac{1.4^3}{3} + 2.8 - \tfrac{0.5^3}{3} - 1\right) \\ &\approx 0.401 \end{aligned}$$

ii
$$\begin{aligned} P(X \geqslant 1) &= \int_1^2 \tfrac{3}{20}(x^2+2)dx \\ &= \tfrac{3}{20}\left[\frac{x^3}{3} + 2x\right]_1^2 \\ &= \tfrac{3}{20}\left(\tfrac{8}{3} + 4 - \tfrac{1}{3} - 2\right) \\ &= 0.65 \end{aligned}$$

c **i**
$$\begin{aligned} \int_0^m \tfrac{3}{20}\left(x^2+2\right)dx &= \tfrac{1}{2} \\ \therefore\ \tfrac{3}{20}\left[\frac{x^3}{3} + 2x\right]_0^m &= \tfrac{1}{2} \\ \therefore\ \tfrac{3}{20}\left(\frac{m^3}{3} + 2m\right) &= \tfrac{1}{2} \\ \therefore\ \frac{m^3}{20} + \frac{3m}{10} &= \tfrac{1}{2} \\ \therefore\ m^3 + 6m - 10 &= 0 \end{aligned}$$

Using technology for $m \in [0, 2]$, we find $m \approx 1.30$.

So, the median ≈ 1.30.

ii
$$\begin{aligned} \mu &= \int_0^2 xf(x)dx \\ &= \int_0^2 \tfrac{3}{20}(x^3+2x)dx \\ &= \tfrac{3}{20}\left[\frac{x^4}{4} + x^2\right]_0^2 \\ &= \tfrac{3}{20}\left(\tfrac{16}{4} + 4\right) \\ &= 1.2 \end{aligned}$$

iii $\mathrm{E}(X^2) = \int_0^2 x^2 f(x)dx$

$$= \int_0^2 \tfrac{3}{20}(x^4 + 2x^2)dx$$

$$= \tfrac{3}{30}\left[\frac{x^5}{5} + \tfrac{2}{3}x^3\right]_0^2$$

$$= \tfrac{3}{20}\left(\tfrac{32}{5} + \tfrac{16}{3}\right)$$

$$= 1.76$$

$\therefore\ \mathrm{Var}(X) = 1.76 - 1.2^2$

$$= 0.32$$

8 From conditional probability,

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

But for independent events,

$$\mathrm{P}(A \mid B) = \mathrm{P}(A)$$

$$\therefore\ \mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

9 **a** The random variable is discrete.

b $\frac{1}{3} + \frac{1}{6} + k + \frac{1}{12} = 1$

$\therefore\ k + \frac{7}{12} = 1$

$\therefore\ k = \frac{5}{12}$

c $\mathrm{E}(X) = -2(\frac{1}{3}) + 0(\frac{1}{6}) + 3(\frac{5}{12}) + 5(\frac{1}{12}) = 1$ and
$\mathrm{Var}(X) = (-2)^2(\frac{1}{3}) + 0^2(\frac{1}{6}) + 3^2(\frac{5}{12}) + 5^2(\frac{1}{12}) - 1^2 = \frac{37}{6}$

$\therefore$ standard deviation $= \sqrt{\frac{37}{6}} \approx 2.48$

d The distribution table is

x	-2	0	3	5
$\mathrm{P}(X = x)$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

The median lies halfway between 0 and 3, so median $= \frac{3}{2}$.
3 is the most probable outcome, so mode $= 3$.

10 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
But A and B are independent, so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\ \mathrm{P}(B)$.

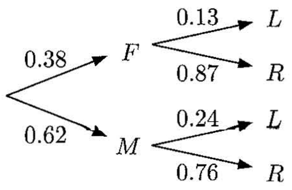
Hence, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$

Let $\mathrm{P}(A) = x$. $\quad\therefore\ 0.63 = x + 0.36 - 0.36x$

$\therefore\ 0.27 = 0.64x$

$\therefore\ x = \mathrm{P}(A) \approx 0.422$

11 The tree diagram uses: M for male $\quad L$ for left handed
F for female $\quad R$ for right handed



a $\mathrm{P}(L) = (0.38)(0.13) + (0.62)(0.24) \approx 0.198$

b $\mathrm{P}(F \mid L) = \dfrac{\mathrm{P}(F \cap L)}{\mathrm{P}(L)} = \dfrac{(0.38)(0.13)}{0.1982} \approx 0.249$

12 $\overline{x} = 80.9$ and $\sum f = 30$

Now $\overline{x} = \dfrac{\sum fx}{\sum f}$, so $\sum fx = 80.9 \times 30 = 2427$

But, $s^2 = \dfrac{\sum (x - \overline{x})^2 f}{\sum f} = \dfrac{\sum x^2 f}{\sum f} - \overline{x}^2$

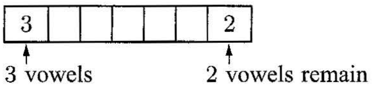
$\therefore\ 296^2 = \dfrac{\sum x^2 f}{30} - 80.9^2$

Thus $\sum x^2 f = 30(296^2 + 80.9^2) \approx 2\,824\,824$

13 **a** The 7 letters are all different.

$\therefore$ there are $7! = 5040$ different words.

b



So, there are $3 \times 2 \times 5! = 720$ ways.

c These 3 vowels can be ordered in 3! ways.
Considering the vowels as one block, this block plus the other 4 can be ordered in 5! ways.

$\therefore$ total number of ways $= 3! \times 5! = 720$

14 **a** The random variable is continuous.

b $\displaystyle\int_0^a \sin(\tfrac{1}{2}x)\ dx = 1$

$\therefore\ \left[-2\cos(\tfrac{1}{2}x)\right]_0^a = 1$

$\therefore\ -2\cos(\frac{a}{2}) + 2 = 1$

$\therefore\ -2\cos(\frac{a}{2}) = -1$

$\therefore\ \cos(\frac{a}{2}) = \frac{1}{2}$

$\therefore\ \frac{a}{2} = \frac{\pi}{3}$

$\therefore\ a = \frac{2\pi}{3}$ {only valid solution as $\sin(\frac{1}{2}x)$ must remain non-negative}

c $\mathrm{E}(X) = \displaystyle\int_0^{\frac{2\pi}{3}} x\sin(\tfrac{1}{2}x)\ dx \approx 1.37$

$\mathrm{Var}(X) = \displaystyle\int_0^{\frac{2\pi}{3}} x^2\sin(\tfrac{1}{2}x)\ dx\ -\ \mathrm{E}(X)^2 \approx 0.248$

$\therefore\ \sigma = \sqrt{\mathrm{Var}(X)} \approx 0.498$

d The median m is the solution of $\displaystyle\int_0^m f(x)\,dx = \tfrac{1}{2}$.

$\therefore\ \displaystyle\int_0^m \sin(\tfrac{1}{2}x)\ dx = \tfrac{1}{2}$

$\therefore\ \left[-2\cos(\tfrac{1}{2}x)\right]_0^m = \frac{1}{2}$

$\therefore\ -2\cos\left(\frac{m}{2}\right) + 2 = \frac{1}{2}$

$\therefore\ 2\cos\left(\frac{m}{2}\right) = 1\frac{1}{2}$

$\therefore\ \cos\left(\frac{m}{2}\right) = \frac{3}{4}$

$\therefore\ \frac{m}{2} \approx 0.7227$

$\therefore\ m \approx 1.45$

$\sin(\frac{1}{2}x)$ on $0 \leqslant x \leqslant \frac{2\pi}{3}$ is a maximum at $x = \frac{2\pi}{3}$

$\therefore$ the modal value of X is $\frac{2\pi}{3}$.

15

die 2	6	5	4	3	2	1	0	
	5	4	3	2	1	0	1	
	4	3	2	1	0	1	2	
	3	2	1	0	1	2	3	
	2	1	0	1	2	3	4	
	1	0	1	2	3	4	5	
		1	2	3	4	5	6	die 1

There are 6 outcomes where the difference is 3.
As all outcomes are equally possible, the probability of the difference being 3 is $\frac{6}{36} = \frac{1}{6}$.

16 $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A \cup B)' = 1 - \frac{1}{12} = \frac{11}{12}$

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\therefore\ \frac{11}{12} = 0.46 + \frac{5}{7} - \mathrm{P}(A \cap B)$

$\therefore\ \mathrm{P}(A \cap B) = 0.46 + \frac{5}{7} - \frac{11}{12}$

$\therefore\ \mathrm{P}(A \cap B) \approx 0.258$

17 **a** X is Poisson.

b mean $m = 12$, variance $= 12$,

standard deviation $= \sqrt{\text{variance}} = \sqrt{12}$.

c $\text{P}(X = 10) = \dfrac{12^{10}e^{-12}}{10!} \approx 0.105$

d $\text{P}(X \geqslant 10) = 1 - \text{P}(X \leqslant 9) \approx 0.758$

18 $X \sim \text{N}(37, 3^2)$.

a X is continuous.

b Since X is continuous the probability of any specified value is zero, in particular $\text{P}(X = 27) = 0$.

Since 27 is more than 3 standard deviations from the mean, it is also highly unlikely to have a score that would be measured as 27.

c As $\dfrac{33-37}{\sqrt{9}} = -\frac{4}{3}$, 33 is $\frac{4}{3}$ standard deviations on the left of the mean.

d $\text{P}(X \geqslant 37) = \text{P}(X \leqslant 37) = \frac{1}{2}$ and since $39 > 37$, $\text{P}(X \geqslant 39) < \frac{1}{2}$.

e $\text{P}(X \geqslant 39) \approx 0.252$

f $\text{P}(31 \leqslant X \leqslant 39) \approx 0.725$

g
$$\begin{aligned}\text{P}(|X-37| \leqslant 2) &= \text{P}(-2 \leqslant X - 37 \leqslant 2)\\ &= \text{P}(35 \leqslant X \leqslant 39)\\ &\approx 0.495\end{aligned}$$

h
$$\begin{aligned}\text{P}(X \geqslant k) &= 0.56\\ \therefore\ \text{P}(X \leqslant k) &= 0.44\\ \therefore\ \text{P}\left(\frac{X-37}{3} \leqslant \frac{k-37}{3}\right) &= 0.44\\ \therefore\ \text{P}\left(Z \leqslant \frac{k-37}{3}\right) &= 0.44\\ \therefore\ \frac{k-37}{3} &\approx -0.151\\ \therefore\ k &\approx 37 + 3(-0.151)\\ \therefore\ k &\approx 36.5\end{aligned}$$

19 **a** Y is binomial.

b $n = 30$ and $p = \frac{1}{5}$

mean $= np = 30 \times \frac{1}{5} = 6$

variance $= np(1-p) = 30 \times \frac{1}{5} \times \frac{4}{5} = 4.8$

standard deviation $= \sqrt{4.8} \approx 2.19$

c $\text{P}(Y = 20) = \binom{30}{20}(\frac{1}{5})^{20}(\frac{4}{5})^{10} \approx 3.38 \times 10^{-8}$

d To get a score of at least $6 + 2(2.191) \approx 10.4$, we need at least 11 correct answers.

$\text{P}(Y \geqslant 11) = 1 - \text{P}(Y \leqslant 10) \approx 0.0256$

20 **a** In general, the number plates are of the type

letter	letter	letter	number	number	number
26	26	26	9	10	10

Total $= (26)^3 \times 9 \times 10^2 = 15\,818\,400$

b $\text{P}(\text{first letter is A}) = \frac{1}{26}$

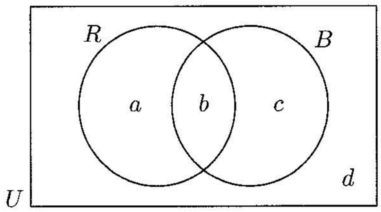
$\text{P}(\text{second letter is B}) = \frac{1}{26}$

$\text{P}(\text{last digit is 0}) = \frac{1}{10}$

$\therefore$ probability of this number plate $= \frac{1}{26} \times \frac{1}{26} \times \frac{1}{10}$

$= \frac{1}{6760}$

21

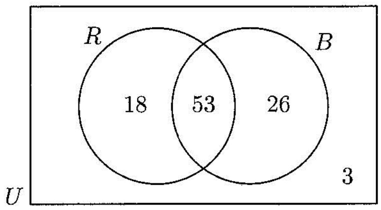


R = right-handed
B = blonde

$$\begin{aligned}a + b &= 71 & \therefore\ b &= 71 - a\\ a + c &= 44 & \therefore\ c &= 44 - a\\ a + d &= 21 & \therefore\ d &= 21 - a\end{aligned}$$

$$\begin{aligned}\text{Now } a + b + c + d &= 100\\ \therefore\ a + (71-a) + (44-a) + (21-a) &= 100\\ \therefore\ 136 - 2a &= 100\\ \therefore\ a &= 18\end{aligned}$$

So, $b = 53$, $c = 26$, $d = 3$



a $\text{P}(R \cap B') = \frac{18}{100} = 0.18$

b $\text{P}(R \cap B) = \frac{53}{100} = 0.53$

c $\text{P}(R \cup B) = \frac{18+53+26}{100} = 0.97$

22
$$\begin{aligned}\text{P}(X \leqslant 15) &= 0.613\\ \therefore\ \text{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{15-\mu}{\sigma}\right) &= 0.613\\ \therefore\ \text{P}\left(Z \leqslant \frac{15-13}{\sigma}\right) &= 0.613\\ \therefore\ \frac{2}{\sigma} &\approx 0.287\\ \therefore\ \sigma &\approx 6.97\end{aligned}$$

23 **a** mean $=$ variance $= \sigma^2 = 9.61$

b $\text{P}(X = x) = \dfrac{9.61^x e^{-9.61}}{x!}$

c **i** $\text{P}(X = 8) = \dfrac{9.61^8 e^{-9.61}}{8!}$

≈ 0.121

ii $\text{P}(X \geqslant 11) = 1 - \text{P}(X \leqslant 10)$

$\approx 1 - 0.632$

≈ 0.368

iii
$$\begin{aligned}&\text{P}(X \geqslant 13 \mid X \geqslant 9)\\ &= \frac{\text{P}(X \geqslant 13 \cap X \geqslant 9)}{\text{P}(X \geqslant 9)}\\ &= \frac{1 - \text{P}(X \leqslant 12)}{1 - \text{P}(X \leqslant 8)}\\ &\approx \frac{1 - 0.827}{1 - 0.378}\\ &\approx 0.278\end{aligned}$$

24
$$\begin{aligned}\text{P}(X \leqslant 24) &= 0.035\\ \therefore\ \text{P}\left(Z \leqslant \frac{24-\mu}{\sigma}\right) &= 0.035\\ \therefore\ \frac{24-\mu}{\sigma} &\approx -1.812\\ \therefore\ 24 - \mu &\approx -1.812\sigma \quad \text{.... (1)}\end{aligned}$$

Also, $\mathrm{P}(X \geqslant 33) = 0.262$

$\therefore\ \mathrm{P}(X \leqslant 33) = 0.738$

$$\therefore\ \mathrm{P}\left(Z \leqslant \frac{33-\mu}{\sigma}\right) = 0.738$$

$$\therefore\ \frac{33-\mu}{\sigma} \approx 0.6372$$

$\therefore\ 33 - \mu \approx 0.6372\sigma$ .... (2)

(1) − (2) gives

$(24-\mu) - (33-\mu) \approx -1.812\sigma - 0.6372\sigma$

$\therefore\ -9 \approx -2.449\sigma$

$\therefore\ \sigma \approx 3.67$

$\therefore\ \mu \approx 24 + 1.812(3.67) \approx 30.7$

25 The committee can be chosen in $\binom{17}{6}$ ways.

a $\mathrm{P}(\text{3 men and 3 women}) = \dfrac{\binom{9}{3}\binom{8}{3}}{\binom{17}{6}} \approx 0.380$

b P(at least two of each sex)

$$= \frac{\binom{9}{2}\binom{8}{4} + \binom{9}{3}\binom{8}{3} + \binom{9}{4}\binom{8}{2}}{\binom{17}{6}} \approx 0.869$$

c P(an even number of women)

$$= \frac{\binom{9}{6}\binom{8}{0} + \binom{9}{4}\binom{8}{2} + \binom{9}{2}\binom{8}{4} + \binom{9}{0}\binom{8}{6}}{\binom{17}{6}} \approx 0.498$$

26 Let X be the number of damaged prawns in the sample. X is binomial with $n = 200$ and $p = 0.03$

mean $= np = 200 \times 0.03 = 6$

standard deviation $= \sqrt{np(1-p)}$

$= \sqrt{200 \times 0.03 \times 0.97}$

≈ 2.41

27 **a** $\mathrm{P}(X=1) + \mathrm{P}(X=2) = \mathrm{P}(X=3)$

$$\therefore\ \frac{m^1e^{-m}}{1!} + \frac{m^2e^{-m}}{2!} = \frac{m^3e^{-m}}{3!}$$

$\therefore\ 6m + 3m^2 = m^3$ $\{\times 6e^m\}$

$\therefore\ m(m^2 - 3m - 6) = 0$

$$\therefore\ m = \frac{3 \pm \sqrt{9 - 4(1)(-6)}}{2}$$

$$\therefore\ m = \frac{3 \pm \sqrt{33}}{2}$$

$$\text{So, } m = \frac{3 + \sqrt{33}}{2} \quad \{\text{as } m > 0\}$$

b $m = \dfrac{3+\sqrt{33}}{2} \approx 4.372$ so $X \sim \mathrm{Po}(4.372)$.

i $\mathrm{P}(X=4) = \dfrac{4.372^4 e^{-4.372}}{4!} \approx 0.192$

ii $\mathrm{P}(X \geqslant 2) = 1 - \mathrm{P}(X \leqslant 1)$

$\approx 1 - 0.068$

≈ 0.932

SOLUTIONS TO TOPIC 6 (CALCULUS)

1 **a** Domain $= \{x \mid x \neq 0,\ x \in \mathbb{R}\}$

b **i** $\displaystyle\lim_{x\to 0^-}\left(\frac{x}{x}\right) = \lim_{x\to 0^-} 1$ $\{\text{since } x \neq 0\}$

$= 1$

ii Similarly, $\displaystyle\lim_{x\to 0^+}\left(\frac{x}{x}\right) = 1.$

c $f(x)$ is not continuous at $x = 0$ because $f(0)$ is not defined.

2 **a** $y = 3 - 2x^2$

$$\therefore\ \frac{dy}{dx} = \lim_{h\to 0}\frac{3 - 2(x+h)^2 - (3 - 2x^2)}{h}$$

$$= \lim_{h\to 0}\frac{3 - 2x^2 - 4xh - 2h^2 - 3 + 2x^2}{h}$$

$$= \lim_{h\to 0}\frac{-2h(2x+h)}{h}$$

$$= \lim_{h\to 0} -2(2x+h) \quad \{\text{as } h \neq 0\}$$

$= -4x$

b When $x = 1$, $y = 3 - 2 = 1$

and $\dfrac{dy}{dx} = -4$

$\therefore$ the normal at $(1, 1)$ has gradient $\frac{1}{4}$

$\therefore$ the equation of the normal is $\dfrac{y-1}{x-1} = \frac{1}{4}$

which is $y - 1 = \frac{1}{4}x - \frac{1}{4}$

or $y = \frac{1}{4}x + \frac{3}{4}$

3 **a** $f(x) = \sqrt{3x^2 + 5x - 2} = (3x^2 + 5x - 2)^{\frac{1}{2}}$

$\therefore\ f'(x) = \frac{1}{2}(3x^2 + 5x - 2)^{-\frac{1}{2}} \times (6x + 5)$

$$= \frac{6x+5}{2\sqrt{3x^2 + 5x - 2}}$$

b $f(x) = x^2e^{2x-3}$ is a product with

$u = x^2$ and $v = e^{2x-3}$

$\therefore\ u' = 2x$ and $v' = 2e^{2x-3}$

$\therefore\ f'(x) = 2xe^{2x-3} + x^2(2e^{2x-3})$

$= 2xe^{2x-3}(1 + x)$

c $f(x) = 3^{x^2 - x - 2}$

$\therefore\ f'(x) = 3^{x^2-x-2}\ln 3 \times (2x - 1)$

d $f(x) = (\sin x)e^{\cos x}$ is a product with

$u = \sin x$ and $v = e^{\cos x}$

$\therefore\ u' = \cos x$ and $v' = -\sin x\, e^{\cos x}$

$\therefore\ f'(x) = \cos x\, e^{\cos x} + \sin x(-\sin x\, e^{\cos x})$

$= e^{\cos x}(\cos x - \sin^2 x)$

4 **a** $\displaystyle\int \frac{3}{x}\,dx = 3\ln|x| + c$

b $\int \cos(3x - 2)\,dx = \frac{1}{3}\sin(3x-2) + c$

c $\displaystyle\int \frac{2x^2 - x - 3}{x^2}\,dx$

$$= \int\left(2 - \frac{1}{x} - 3x^{-2}\right)dx$$

$$= 2x - \ln|x| - \frac{3x^{-1}}{-1} + c$$

$$= 2x - \ln|x| + \frac{3}{x} + c$$

5 **a** $y = \ln(2x^2 + 8)$

$$\therefore\ \frac{dy}{dx} = \frac{4x}{2x^2 + 8} = \frac{2x}{x^2 + 4}$$

b $y = \dfrac{x+2}{x^2+3}$ is a quotient with

$u = x + 2$ and $v = x^2 + 3$

$\therefore\ u' = 1$ and $v' = 2x$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \frac{1(x^2+3)-(x+2)2x}{(x^2+3)^2}\\ &= \frac{x^2+3-2x^2-4x}{(x^2+3)^2}\\ &= \frac{3-4x-x^2}{(x^2+3)^2}\end{aligned}$$

c $y = \arcsin(2x)$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-(2x)^2}} \times 2\\ &= \frac{2}{\sqrt{1-4x^2}}\end{aligned}$$

d $y = e^{x\ln x}$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= e^{x\ln x} \times \left[1(\ln x) + x\left(\frac{1}{x}\right)\right]\\ &\qquad \{\text{chain and product rule}\}\\ &= e^{x\ln x}(\ln x + 1)\end{aligned}$$

6 $$\begin{aligned}\int_3^5 \frac{x}{x^2-8}\,dx &= \tfrac{1}{2}\int_3^5 \frac{2x}{x^2-8}\,dx\\ &= \tfrac{1}{2}\left[\ln\left|x^2-8\right|\right]_3^5\\ &= \tfrac{1}{2}(\ln 17 - \ln 1)\\ &= \tfrac{1}{2}\ln 17\end{aligned}$$

7 $f(x) = \dfrac{x-4}{x+2}$

Since $f(3) = \frac{-1}{5}$, the point of contact is $\left(3, -\frac{1}{5}\right)$.

Now $$\begin{aligned}f'(x) &= \frac{1(x+2)-(x-4)1}{(x+2)^2} \quad \{\text{quotient rule}\}\\ &= \frac{6}{(x+2)^2}\end{aligned}$$

$\therefore\ f'(3) = \frac{6}{25}$

$\therefore$ the tangent has equation

$$\begin{aligned}6x - 25y &= 6(3) - 25\left(-\tfrac{1}{5}\right)\\ \therefore\ 6x - 25y &= 23\end{aligned}$$

8 **a** The curves meet where

$$\begin{aligned}x(x+4)(x-4) &= 9x\\ \therefore\ x(x^2-16) &= 9x\\ \therefore\ x(x^2-25) &= 0\\ \therefore\ x(x+5)(x-5) &= 0\\ \therefore\ x &= 0 \text{ or } \pm 5\end{aligned}$$

$\therefore$ the curves meet at $(-5, -45)$, $(0, 0)$, and $(5, 45)$.

b **i** Area A $= \displaystyle\int_{-5}^{0} (x(x+4)(x-4) - 9x)\,dx$

ii Area B $= \displaystyle\int_{0}^{5} (9x - x(x+4)(x-4))\,dx$

iii Area A + Area B

$$= \int_{-5}^{5} |x(x+4)(x-4) - 9x|\,dx$$

9 $$\begin{aligned}\int \sin^2 3x\,dx &= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 6x\right)dx\\ &= \tfrac{1}{2}x - \tfrac{1}{2}\left(\tfrac{1}{6}\right)\sin 6x + c\\ &= \tfrac{1}{2}x - \tfrac{1}{12}\sin 6x + c\end{aligned}$$

10 **a** $$\begin{aligned}y &= \frac{3}{x^2} = 3x^{-2}\\ \therefore\ \frac{dy}{dx} &= -6x^{-3}\\ \therefore\ \frac{d^2y}{dx^2} &= 18x^{-4} = \frac{18}{x^4}\end{aligned}$$

b $y = x^2 \sin 3x$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= 2x\sin 3x + x^2(3\cos 3x) \quad \{\text{product rule}\}\\ &= 2x\sin 3x + 3x^2\cos 3x\\ \therefore\ \frac{d^2y}{dx^2} &= 2\sin 3x + 2x(3\cos 3x) + 6x\cos 3x\\ &\qquad + 3x^2(-3\sin 3x)\\ &= 12x\cos 3x + (2-9x^2)\sin 3x\end{aligned}$$

11 $$\begin{aligned}\int_0^k \frac{x}{\sqrt{x^2+4}}\,dx &= 1\\ \therefore\ \tfrac{1}{2}\int_0^k (x^2+4)^{-\frac{1}{2}}(2x)\,dx &= 1\\ \therefore\ \tfrac{1}{2}\left[\frac{(x^2+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^k &= 1\\ \therefore\ \left[\sqrt{x^2+4}\right]_0^k &= 1\\ \therefore\ \sqrt{k^2+4} - 2 &= 1\\ \therefore\ \sqrt{k^2+4} &= 3\\ \therefore\ k^2 + 4 &= 9\\ \therefore\ k^2 &= 5\\ \therefore\ k &= \pm\sqrt{5}\end{aligned}$$

But $k > 0$, so $k = \sqrt{5}$.

12 For $\int x\ln x\,dx$ we integrate by parts with:

$u = \ln x \qquad v' = x$

$u' = \dfrac{1}{x} \qquad v = \dfrac{x^2}{2}$

$$\begin{aligned}\therefore\ \int x\ln x\,dx &= (\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)dx\\ &= \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx\\ &= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c\end{aligned}$$

13 $$\begin{aligned}y &= x^2 - 2x - 4\\ \therefore\ \frac{dy}{dx} &= 2x - 2\end{aligned}$$

When $x = -1$, $y = (-1)^2 - 2(-1) - 4$
$= -1$

and $\dfrac{dy}{dx} = 2(-1) - 2$
$= -4$

So, the normal has gradient $\frac{1}{4}$.

Thus, the equation of the normal is

$$\begin{aligned}x - 4y &= (-1) - 4(-1)\\ \therefore\ x - 4y &= 3\end{aligned}$$

14 **a** $f(x) = \dfrac{x}{x^2-2}$

$\therefore$ $f(x)$ is undefined when $x^2 - 2 = 0$

$\therefore\ x = \pm\sqrt{2}$

b $$\begin{aligned}f'(x) &= \frac{1(x^2-2) - x(2x)}{(x^2-2)^2}\\ &= \frac{-(x^2+2)}{(x^2-2)^2}\end{aligned}$$

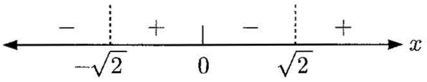
$\therefore$ $f'(x) < 0$ for all x for which $f'(x)$ is defined, so $f(x)$ is never increasing.

c $$f'(x) = \frac{-x^2-2}{(x^2-2)^2}$$

$$\therefore\ f''(x) = \frac{-2x(x^2-2)^2 - (-x^2-2)\left[2(x^2-2)(2x)\right]}{(x^2-2)^4}$$

$$= \frac{-2x(x^2-2)\left[(x^2-2)+2(-x^2-2)\right]}{(x^2-2)^4}$$

$$= \frac{2x(x^2+6)}{(x^2-2)^3}$$



$\therefore$ $f(x)$ is concave down for $x < -\sqrt{2}$ and $0 \leqslant x < \sqrt{2}$.

15 $v = t^3 - 3t^2e^{0.05t}$

$\therefore$ the distance travelled in the first 5 seconds

$$= \int_0^5 |v|\ dt$$

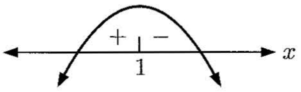
$$= \int_0^5 \left|t^3 - 3t^2e^{0.05t}\right|\ dt \approx 28.2 \text{ units} \quad \{\text{using technology}\}$$

16 **a** $y = xe^{-x}$

$$\therefore\ \frac{dy}{dx} = 1 \times e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\}$$

$$= e^{-x}(1-x)$$

$\therefore$ there is a stationary point when $x = 1$.



When $x = 1$, $y = 1e^{-1} = \frac{1}{e}$.

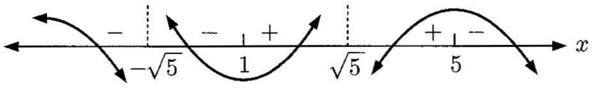
$\therefore$ there is a local maximum at $\left(1, \frac{1}{e}\right)$.

b $$y = \frac{x-3}{x^2-5}$$

$$\therefore\ \frac{dy}{dx} = \frac{1(x^2-5)-(x-3)2x}{(x^2-5)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-x^2+6x-5}{(x^2-5)^2}$$

$$= \frac{-(x-1)(x-5)}{(x^2-5)^2}$$



When $x = 1$, $y = \frac{-2}{-4} = \frac{1}{2}$.

When $x = 5$, $y = \frac{2}{20} = \frac{1}{10}$

$\therefore$ there is a local minimum at $(1, \frac{1}{2})$ and a local maximum at $(5, \frac{1}{10})$.

17 $$\int \tan^2 2x\,dx = \int \left(\sec^2 2x - 1\right) dx$$

$$= \tfrac{1}{2}\tan 2x - x + c$$

18 Let $f(x) = x^3 + 2x + 1$

$$\therefore\ f(-1) = (-1)^3 + 2(-1) + 1$$

$$= -2$$

Now $f'(x) = 3x^2 + 2$

$$\therefore\ f'(-1) = 3(-1)^2 + 2$$

$$= 5$$

$\therefore$ the tangent at $(-1, -2)$ has gradient 5, and its equation is

$$5x - y = 5(-1) - (-2)$$

$$\therefore\ 5x - y = -3 \quad \text{or} \quad y = 5x + 3$$

Now $y = 5x + 3$ meets $y = x^3 + 2x + 1$ where

$$x^3 + 2x + 1 = 5x + 3$$

$$\therefore\ x^3 - 3x - 2 = 0$$

The tangent touches the curve at $x = -1$, so $(x+1)^2$ is a factor of this cubic.

$$\therefore\ (x+1)^2(x-2) = 0$$

$$\therefore\ x = -1 \text{ or } 2$$

When $x = 2$, $y = 2^3 + 2(2) + 1 = 13$

So, the tangent meets the curve again at $(2, 13)$.

19 **a** $y = ax^2 + bx + c$

$$\therefore\ \frac{dy}{dx}$$

$$= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2+bx+c)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{ax^2} + 2axh + ah^2 + \cancel{bx} + bh + \cancel{c} - \cancel{ax^2} - \cancel{bx} - \cancel{c}}{h}$$

$$= \lim_{h\to 0} \frac{h(2ax + ah + b)}{h}$$

$$= \lim_{h\to 0} (2ax + ah + b) \quad \{\text{since } h \neq 0\}$$

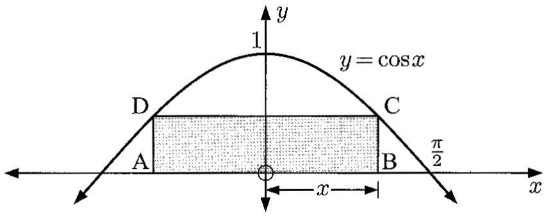
$$= 2ax + b$$

b At the vertex of the quadratic, the gradient is 0

$$\therefore\ 2ax + b = 0$$

$$\therefore\ x = -\frac{b}{2a}$$

20



Let x be the x-coordinate of C.

Then C has coordinates $(x, \cos x)$.

So rectangle ABCD has area $A = 2x\cos x$

$\therefore$ $A' = 2\cos x + 2x(-\sin x)$ {product rule}

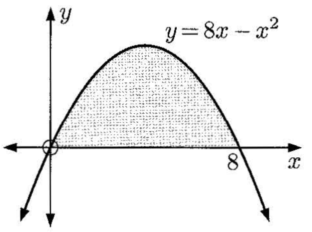
$\therefore$ $A' = 0$ when $2x\sin x = 2\cos x$

$$\therefore\ x\tan x = 1$$

$$\therefore\ x \approx 0.860 \quad \{\text{technology}\}$$

$\therefore$ C has coordinates $(0.860, 0.652)$

21 $y = 8x - x^2 = x(8-x)$ cuts the x-axis at 0 and 8



$$\text{area} = \int_0^8 \left(8x - x^2\right) dx$$

$$= \left[\frac{8x^2}{2} - \frac{x^3}{3}\right]_0^8$$

$$= \left(256 - \tfrac{512}{3}\right) - 0$$

$$= \tfrac{256}{3} \text{ units}^2$$

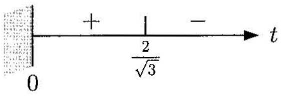
22 **a** $$s(t) = 12t - 3t^3 + 1$$

$$\therefore\ v(t) = s'(t) = 12 - 9t^2$$

$$\therefore\ a(t) = v'(t) = -18t$$

b $v(t) = 12 - 9t^2$
$= 3(4 - 3t^2)$
$= 3(2 + t\sqrt{3})(2 - t\sqrt{3})$

which has sign diagram



$a(t) = -18t$

which has sign diagram

i Speed is decreasing when $v(t)$ and $a(t)$ have the opposite sign.
$\therefore$ speed is decreasing for $0 \leqslant t \leqslant \frac{2}{\sqrt{3}}$.

ii Velocity is decreasing when $v'(t) \leqslant 0$
Since $v'(t) = a(t) \leqslant 0$ for all $t \geqslant 0$, the velocity is decreasing for $t \geqslant 0$.

23 For $\int \arctan x \, dx$ we integrate by parts with:

$u = \arctan x \qquad v' = 1$

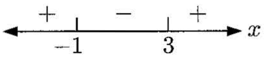
$u' = \dfrac{1}{1+x^2} \qquad v = x$

$\therefore \int \arctan x \, dx$

$$= x \arctan x - \int \frac{x}{1+x^2}\, dx$$
$$= x \arctan x - \tfrac{1}{2}\int \frac{2x}{1+x^2}\, dx$$
$$= x \arctan x - \tfrac{1}{2}\ln\left|1+x^2\right| + c$$
$$= x \arctan x - \tfrac{1}{2}\ln(1+x^2) + c \quad \{\text{since } 1+x^2 > 0\}$$

24 **a** $f(x) = x^3 - 3x^2 - 9x + 5$
$\therefore f'(x) = 3x^2 - 6x - 9$
$= 3(x+1)(x-3)$

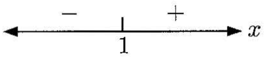
which has sign diagram



So, $f(x)$ is increasing for $x \leqslant -1$ and $x \geqslant 3$.

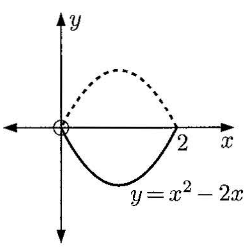
b $f'(x) = 3x^2 - 6x - 9$
$\therefore f''(x) = 6x - 6$
$= 6(x-1)$

which has sign diagram



So, $f(x)$ is concave up for $x \geqslant 1$.

25



$$V = \pi \int_0^2 y^2 \, dx$$
$$= \pi \int_0^2 (x^2 - 2x)^2 \, dx$$
$$= \pi \int_0^2 x^4 - 4x^3 + 4x^2 \, dx$$
$$= \pi \left[\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}\right]_0^2$$
$$= \pi\left(\tfrac{32}{5} - 16 + \tfrac{32}{3}\right)$$
$$= \tfrac{16}{15}\pi \text{ units}^3$$

26 $$\int_0^4 \frac{1}{\sqrt{x+4}}\, dx = \int_0^4 (x+4)^{-\frac{1}{2}}\, dx$$
$$= \left[\frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4$$
$$= \left[2\sqrt{x+4}\right]_0^4$$
$$= 2\sqrt{8} - 2\sqrt{4}$$
$$= 4\sqrt{2} - 4$$

27 **a** $$x^2 - xy^2 + y = 21$$
$$\therefore \frac{d}{dx}(x^2) - \frac{d}{dx}(xy^2) + \frac{d}{dx}(y) = \frac{d}{dx}(21)$$
$$\therefore 2x - \left[y^2 + x\left(2y\frac{dy}{dx}\right)\right] + \frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx}(1 - 2xy) = y^2 - 2x$$
$$\therefore \frac{dy}{dx} = \frac{y^2 - 2x}{1 - 2xy}$$

b $$e^y \sin 2x = 1$$
$$\therefore \frac{d}{dx}(e^y \sin 2x) = \frac{d}{dx}(1)$$
$$\therefore e^y \frac{dy}{dx}\sin 2x + e^y(2\cos 2x) = 0$$
$$\therefore \frac{dy}{dx} = \frac{-2e^y \cos 2x}{e^y \sin 2x}$$
$$= -2\cot 2x$$

28 **a** $$\int xe^{-x^2} dx = -\tfrac{1}{2}\int (-2x)e^{-x^2}\, dx$$
$$= -\tfrac{1}{2}\int e^u \frac{du}{dx}\, dx \quad \left\{u = -x^2,\ \frac{du}{dx} = -2x\right\}$$
$$= -\tfrac{1}{2}\int e^u \, du$$
$$= -\tfrac{1}{2}e^u + c$$
$$= -\tfrac{1}{2}e^{-x^2} + c$$

b $$\int 5^x dx = \frac{1}{\ln 5}\int 5^x \ln 5 \, dx$$
$$= \frac{1}{\ln 5} \times 5^x + c$$
$$= \frac{5^x}{\ln 5} + c$$

c $$\int (x^2+1)^3\, dx = \int x^6 + 3x^4 + 3x^2 + 1\, dx$$
$$= \tfrac{1}{7}x^7 + \tfrac{3}{5}x^5 + x^3 + x + c$$

29 $$\int_0^k \sin x \, dx = 0.42$$
$$\therefore [-\cos x]_0^k = 0.42$$
$$\therefore -\cos k + \cos 0 = 0.42$$
$$\therefore \cos k = 0.58$$
$$\therefore k \approx 0.952 \quad \{0 \leqslant k \leqslant \pi\}$$

30 $K = \frac{1}{2}mv^2$

Differentiating with respect to t gives

$$\frac{dK}{dt} = \tfrac{1}{2}\left[\frac{dm}{dt}v^2 + m\left(2v\frac{dv}{dt}\right)\right]$$
$$\therefore 50\,000 = \tfrac{1}{2}\left(-10v^2 + 2mv\frac{dv}{dt}\right)$$

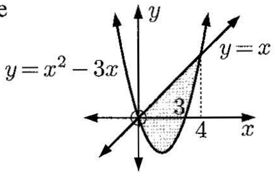
Particular case: When $m = 4000$, $v = 8$:

$$50\,000 = \tfrac{1}{2}\left(-10(64) + 2(4000)(8)\frac{dv}{dt}\right)$$

$$\therefore\ 100\,000 = -640 + 64\,000\,\frac{dv}{dt}$$
$$\therefore\ 100\,640 = 64\,000\,\frac{dv}{dt}$$
$$\therefore\ \frac{dv}{dt} \approx 1.57$$

So, the velocity is increasing at 1.57 km s^{-2}.

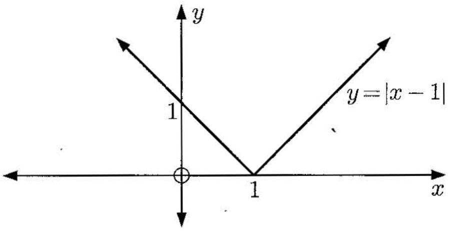
31 $y = x^2 - 3x$ meets $y = x$ where



$$x^2 - 3x = x$$
$$\therefore\ x^2 - 4x = 0$$
$$\therefore\ x(x-4) = 0$$
$$\therefore\ x = 0 \text{ or } 4$$

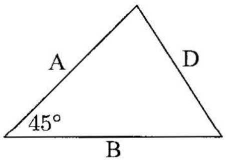
$$\text{area} = \int_0^4 \left(x - (x^2 - 3x)\right) dx = \int_0^4 \left(4x - x^2\right) dx$$
$$= \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4$$
$$= 32 - \tfrac{64}{3}$$
$$= \tfrac{32}{3} \text{ units}^2$$

32



$$\therefore\ \int_0^3 |x-1|\,dx = \int_0^1 (1-x)\,dx + \int_1^3 (x-1)\,dx$$
$$= \left[x - \tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{2}x^2 - x\right]_1^3$$
$$= \left(1 - \tfrac{1}{2}\times 1^2\right) - \left(0 - \tfrac{1}{2}\times 0^2\right)$$
$$+ \left(\tfrac{1}{2}\times 3^2 - 3\right) - \left(\tfrac{1}{2}\times 1^2 - 1\right)$$
$$= \tfrac{5}{2}$$

33



Let A and B be the distances travelled by the 28 km h^{-1} and the 32 km h^{-1} roadrunners respectively.

$$\therefore\ D^2 = A^2 + B^2 - 2AB\cos 45°$$
$$\therefore\ D^2 = A^2 + B^2 - \sqrt{2}AB$$

Differentiating with respect to t gives

$$2D\frac{dD}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt} - \sqrt{2}\left(\frac{dA}{dt}B + A\frac{dB}{dt}\right)$$
$$\therefore\ 2D\frac{dD}{dt} = 2A(28) + 2B(32) - \sqrt{2}(28B + 32A)$$

Particular case:

After 15 minutes or $\frac{1}{4}$ hour $A = 7$ and $B = 8$, and so

$$D^2 = 7^2 + 8^2 - \sqrt{2}(7)(8)$$
$$\therefore\ D = \sqrt{113 - 56\sqrt{2}} \approx 5.814$$
$$\therefore\ 2(5.814)\frac{dD}{dt} = 56(7) + 64(8) - \sqrt{2}\,[28(8) + 32(7)]$$
$$\therefore\ 11.628\,\frac{dD}{dt} = 904 - 448\sqrt{2}$$
$$\therefore\ \frac{dD}{dt} \approx 23.3$$

So, the distance between the roadrunners is increasing at a rate of 23.3 km h^{-1}.

34 **a**

$$\frac{d}{dx}\left(\ln|\sec x + \tan x|\right)$$
$$= \frac{1}{\sec x + \tan x} \times \frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \sec x$$
$$\therefore\ \int \sec x\,dx = \ln|\sec x + \tan x| + c$$

b $\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx$

Let $u = \sec x$ $\quad v' = \sec^2 x$

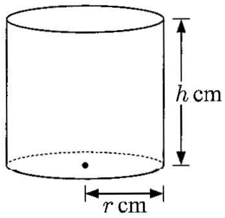
$u' = \sec x \tan x$ $\quad v = \tan x$

$$\therefore\ \int \sec^3 x\,dx$$
$$= \sec x \tan x - \int \sec x \tan x \tan x\,dx$$
$$= \sec x \tan x - \int \frac{\sin^2 x}{\cos^3 x}\,dx$$
$$= \sec x \tan x - \int \frac{1 - \cos^2 x}{\cos^3 x}\,dx$$
$$= \sec x \tan x - \int \frac{1}{\cos^3 x} - \frac{1}{\cos x}\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$$

{using **a**}

$$\therefore\ \int \sec^3 x = \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + c$$

35 The bin has capacity 500 litres



$$= 500\,000 \text{ mL}$$
$$\therefore\ \pi r^2 h = 500\,000$$
$$\therefore\ h = \frac{500\,000}{\pi r^2}$$

Surface area $A = 2\pi rh + \pi r^2$

$$= 2\pi r\left(\frac{500\,000}{\pi r^2}\right) + \pi r^2$$
$$= 1\,000\,000 r^{-1} + \pi r^2$$
$$\therefore\ \frac{dA}{dr} = -\frac{1\,000\,000}{r^2} + 2\pi r$$
$$\therefore\ \frac{dA}{dr} = 0 \text{ when } 2\pi r = \frac{1\,000\,000}{r^2}$$
$$\therefore\ 2\pi r^3 = 1\,000\,000$$
$$\therefore\ r = \sqrt[3]{\frac{1\,000\,000}{2\pi}} \approx 54.2$$
$$\text{and } h \approx \frac{500\,000}{\pi(54.2)^2} \approx 54.2$$

So, the surface area of the bin is minimised when the bin has a base radius and height of 54.2 cm.

36 Let $u = 3x - 4$

$$\therefore \quad x = \frac{u+4}{3} \quad \text{and} \quad \frac{du}{dx} = 3.$$

$$\begin{aligned}\therefore \quad \int x\sqrt{3x-4}\,dx &= \tfrac{1}{3}\int \frac{u+4}{3}\sqrt{u}(3)\,dx \\ &= \tfrac{1}{9}\int \left((u+4)\sqrt{u}\right)du \\ &= \tfrac{1}{9}\int (u^{\frac{3}{2}} + 4u^{\frac{1}{2}})\,du \\ &= \tfrac{1}{9}\left(\tfrac{2}{5}u^{\frac{5}{2}} + \tfrac{8}{3}u^{\frac{3}{2}}\right) + c \\ &= \tfrac{2}{45}(3x-4)^{\frac{5}{2}} + \tfrac{8}{27}(3x-4)^{\frac{3}{2}} + c\end{aligned}$$

37 **a**
$$\begin{aligned}\frac{x+2}{(x-1)(x+3)} &= \frac{A}{x-1} + \frac{B}{x+3} \\ &= \frac{A(x+3)+B(x-1)}{(x-1)(x+3)} \\ &= \frac{(A+B)x+3A-B}{(x-1)(x+3)}\end{aligned}$$

$\therefore \quad A + B = 1 \quad .... (1)$

$3A - B = 2 \quad .... (2)$

From (1), $B = 1 - A$

Substituting into (2), $3A - (1 - A) = 2$

$\therefore \quad 4A = 3$

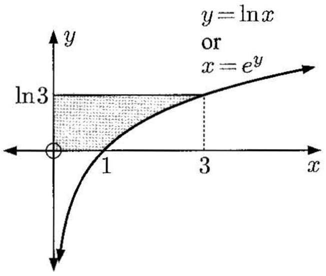
$\therefore \quad A = \frac{3}{4}$

$\therefore \quad B = \frac{1}{4}$

$$\therefore \quad \frac{x+2}{(x-1)(x+3)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+3}$$

b
$$\begin{aligned}\int \frac{x+2}{(x-1)(x+3)}\,dx &= \int \left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{4}}{x+3}\right)dx \\ &= \tfrac{3}{4}\ln|x-1| + \tfrac{1}{4}\ln|x+3| + c\end{aligned}$$

38



$$\begin{aligned}V &= \pi\int_0^{\ln 3} x^2\,dy \\ &= \pi\int_0^{\ln 3} e^{2y}\,dy \\ &= \pi\left[\tfrac{1}{2}e^{2y}\right]_0^{\ln 3} \\ &= \tfrac{1}{2}\pi(e^{2\ln 3} - e^0) \\ &= \tfrac{1}{2}\pi(e^{\ln 3^2} - 1) \\ &= \tfrac{1}{2}\pi(9-1) \\ &= 4\pi \text{ units}^3\end{aligned}$$

39 Let $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$

When $x = 1$, $\theta = \frac{\pi}{6}$

When $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

$$\begin{aligned}\therefore \quad \int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{2\cos\theta}{2\cos\theta}\,d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \\ &= \tfrac{\pi}{3} - \tfrac{\pi}{6} \\ &= \tfrac{\pi}{6}\end{aligned}$$

40
$$\begin{aligned}\int x\tan^2 x\,dx &= \int x(\sec^2 x - 1)\,dx \\ &= \int x\sec^2 x - x\,dx \\ &= \frac{-x^2}{2} + \int x\sec^2 x\,dx\end{aligned}$$

We integrate by parts with $u = x \quad v' = \sec^2 x$
$u' = 1 \quad v = \tan x$

$$\begin{aligned}\therefore \quad \int x\tan^2 x\,dx &= \frac{-x^2}{2} + x\tan x - \int \tan x\,dx \\ &= \frac{-x^2}{2} + x\tan x + \int \frac{-\sin x}{\cos x}\,dx \\ &= \frac{-x^2}{2} + x\tan x + \ln|\cos x| + c\end{aligned}$$

41
$$\int_0^a \frac{x^2}{x^3+1}\,dx = 2$$

$$\therefore \quad \tfrac{1}{3}\int_0^a \frac{3x^2}{x^3+1}\,dx = 2$$

$$\therefore \quad \int_0^a \frac{3x^2}{x^3+1}\,dx = 6$$

$$\therefore \quad \left[\ln\left|x^3+1\right|\right]_0^a = 6$$

$$\therefore \quad \ln\left|a^3+1\right| - \ln 1 = 6$$

$$\therefore \quad \ln\left|a^3+1\right| = 6$$

$$\therefore \quad a^3 + 1 = e^6 \quad \{a > 0\}$$

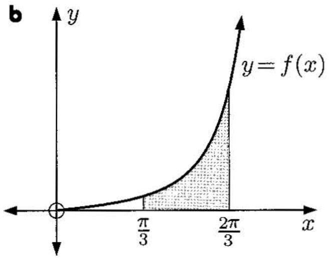
$$\therefore \quad a^3 = e^6 - 1$$

$$\therefore \quad a = \sqrt[3]{e^6 - 1}$$

42 **a** Let $u = 1 + \cos x$, $\dfrac{du}{dx} = -\sin x$.

$$\begin{aligned}\therefore \quad \int \frac{\sin x}{(1+\cos x)^2}\,dx &= \int u^{-2}\left(-\frac{du}{dx}\right)dx \\ &= \int -u^{-2}\,du \\ &= \frac{-u^{-1}}{-1} + c \\ &= \frac{1}{u} + c \\ &= \frac{1}{1+\cos x} + c\end{aligned}$$

b



Area

$$\begin{aligned}&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{(1+\cos x)^2}\,dx \\ &= \left[\frac{1}{1+\cos x}\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\ &= \frac{1}{1-\frac{1}{2}} - \frac{1}{1+\frac{1}{2}} \\ &= 2 - \tfrac{2}{3} \\ &= \tfrac{4}{3} \text{ units}^2\end{aligned}$$

43
$$\begin{aligned}\int \frac{\sin^3 x}{\cos^2 x}\,dx &= \int \frac{\sin^2 x\sin x}{\cos^2 x}\,dx \\ &= \int \frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,dx \\ &= \int \left([\cos x]^{-2}\sin x - \sin x\right)dx\end{aligned}$$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$.

$$\therefore \int \frac{\sin^3 x}{\cos^2 x}\,dx = \int u^{-2}\left(-\frac{du}{dx}\right)dx - (-\cos x) + c$$
$$= -\frac{u^{-1}}{-1} + \cos x + c$$
$$= \frac{1}{u} + \cos x + c$$
$$= \sec x + \cos x + c$$

44 $y = \dfrac{\ln x}{x^2}$ $\quad\therefore\ \dfrac{dy}{dx} = \dfrac{\frac{1}{x}(x^2) - \ln x(2x)}{x^4}$

$$= \frac{x - 2x\ln x}{x^4}$$
$$= \frac{1 - 2\ln x}{x^3}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{\left(-\frac{2}{x}\right)x^3 - (1 - 2\ln x)3x^2}{x^6}$$
$$= \frac{-2x^2 - 3x^2 + 6x^2\ln x}{x^6}$$
$$= \frac{-5 + 6\ln x}{x^4}$$

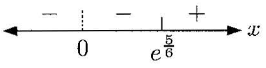
$\therefore \dfrac{d^2y}{dx^2} = 0$ when $-5 + 6\ln x = 0$

$\therefore\ 6\ln x = 5$

$\therefore\ \ln x = \frac{5}{6}$

$\therefore\ x = e^{\frac{5}{6}}$

$f''(x)$ has sign diagram:



There is a point of inflection at $\left(e^{\frac{5}{6}}, \dfrac{5}{6e^{\frac{5}{3}}}\right)$.

45 **a** Let $y = x^{\frac{1}{x}}$, so $\ln y = \dfrac{1}{x}\ln x = \dfrac{\ln x}{x}$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
$$\therefore \frac{dy}{dx} = \left(\frac{1 - \ln x}{x^2}\right)x^{\frac{1}{x}}$$

b A stationary point exists when $\dfrac{dy}{dx} = 0$

$\therefore\ \ln x = 1\ \ \therefore\ x = e$

When $x = e$, $y = e^{\frac{1}{e}}$

$\therefore\ (e, e^{\frac{1}{e}})$ is a stationary point.

46 We integrate by parts with $u = x^2 \quad v' = \sin x$
$u' = 2x \quad v = -\cos x$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx$$
$$= -x^2\cos x + 2\int x\cos x\,dx$$

We now integrate by parts with $u = x \quad v' = \cos x$
$u' = 1 \quad v = \sin x$

$$\therefore \int x^2\sin x\,dx = -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right)$$
$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

47 $$\int \frac{1 - 2x}{\sqrt{1 - x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{1 - x^2}}\,dx + \int \frac{-2x}{\sqrt{1 - x^2}}\,dx$$
$$= \arcsin x + \int \frac{1}{\sqrt{u}}\frac{du}{dx}\,dx$$

{where $u = 1 - x^2$, $\dfrac{du}{dx} = -2x$}

$$= \arcsin x + \int u^{-\frac{1}{2}}\,du$$
$$= \arcsin x + \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \arcsin x + 2\sqrt{1 - x^2} + c$$

48 $f(x) = 2xe^x - 6e^x - 3x^2 + 12x + 5$

$\therefore\ f'(x) = 2e^x + 2xe^x - 6e^x - 6x + 12$
$= e^x(2x - 4) - 6x + 12$
$= 2e^x(x - 2) - 6(x - 2)$
$= 2(x - 2)(e^x - 3)$

$\therefore\ f'(x) = 0$ when $x = 2$ or $e^x = 3$

$\therefore\ x = 2$ or $\ln 3$

49 We integrate by parts with $u = \arccos x \quad v' = 1$
$u' = \dfrac{-1}{\sqrt{1 - x^2}} \quad v = x$

$$\therefore \int \arccos x\,dx = \int 1\arccos x\,dx$$
$$= x\arccos x - \int \frac{-x}{\sqrt{1 - x^2}}\,dx$$

Let $w = 1 - x^2$, $\dfrac{dw}{dx} = -2x$

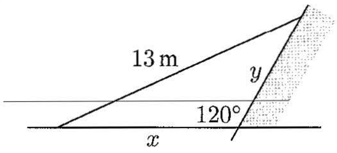
$$\therefore \int \arccos x\,dx = x\arccos x - \int \frac{1}{\sqrt{w}}\left(\tfrac{1}{2}\frac{dw}{dx}\right)dx$$
$$= x\arccos x - \tfrac{1}{2}\int w^{-\frac{1}{2}}\,dw$$
$$= x\arccos x - \tfrac{1}{2}(2w^{\frac{1}{2}}) + c$$
$$= x\arccos x - \sqrt{1 - x^2} + c$$

50 Let $u = 1 + \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = 2$. When $x = \frac{\pi}{2}$, $u = 1$.

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{u}\left(-\frac{du}{dx}\right)dx$$
$$= -\int_2^1 \frac{1}{u}\,du$$
$$= \int_1^2 \frac{1}{u}\,du$$
$$= \big[\ln|u|\big]_1^2$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

51



$\dfrac{dx}{dt} = 2\ \text{m sec}^{-1}$

By the cosine rule,

$13^2 = x^2 + y^2 - 2xy\cos(120°)$

$\therefore\ 169 = x^2 + y^2 + xy$

$$\therefore \quad 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} + \frac{dx}{dt}\,y + x\,\frac{dy}{dt} = 0$$

$$\therefore \quad (2x+y)\,\frac{dx}{dt} + (x+2y)\,\frac{dy}{dt} = 0$$

Particular case: $x = 7$

$$\therefore \quad 169 = 7^2 + y^2 + 7y$$

$$\therefore \quad y^2 + 7y - 120 = 0$$

$$\therefore \quad (y-8)(y+15) = 0$$

$$\therefore \quad y = 8 \text{ or } -15$$

$$\therefore \quad y = 8 \quad \{y > 0\}$$

$$\therefore \quad 22(2) + 23\,\frac{dy}{dt} = 0 \text{ and so } \frac{dy}{dt} = -\tfrac{44}{23}$$

$\therefore$ at that instant, the ladder is moving down the wall at $\frac{44}{23}\ \text{m sec}^{-1}$.

52 Let $u = \arctan x$, $\dfrac{du}{dx} = \dfrac{1}{1+x^2}$.

$$\therefore \quad \int \frac{\arctan x}{1+x^2}\,dx = \int u\,\frac{du}{dx}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + c$$

$$= \tfrac{1}{2}(\arctan x)^2 + c$$

53 Let $x = 2\sin\theta$, $\dfrac{dx}{d\theta} = 2\cos\theta$.

$$\therefore \quad \int \sqrt{4-x^2}\,dx$$

$$= \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= \int 2\sqrt{1-\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta$$

$$= 4\int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

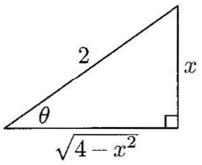
$$= 4\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + c$$

$$= 2\theta + \sin 2\theta + c$$

$$= 2\theta + 2\sin\theta\cos\theta + c$$

$$= 2\arcsin\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + c$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \tfrac{1}{2}x\sqrt{4-x^2} + c$$



SOLUTIONS TO EXAMINATION PRACTICE SET 1

1 **a** $\displaystyle\sum_{r=1}^{3} (2r + 2^r) = (2+2) + (4+4) + (6+8) = 26$

b $\displaystyle\sum_{r=1}^{n} (2r + 2^r)$

$$= 2 + 2^1 + 4 + 2^2 + 6 + 2^3 + 8 + 2^4 + \dots + 2n + 2^n$$

$$= (2 + 4 + 6 + 8 + \dots + 2n) + (2 + 2^2 + 2^4 + \dots + 2^n)$$

$$= \underbrace{\frac{n}{2}(2+2n)}_{\text{arithmetic sum}} + \underbrace{\frac{2(2^n-1)}{2-1}}_{\text{geometric sum}}$$

$$= n(n+1) + 2(2^n - 1)$$

2 $\log_a \sqrt{72} = \log_a (2^3 3^2)^{\frac{1}{2}}$

$$= \log_a (2^{\frac{3}{2}}\,3^1)$$

$$= \tfrac{3}{2}\log_a 2 + \log_a 3$$

$$= \tfrac{3}{2}b + c$$

3

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\cos^2 x + \tan^2 x\right)\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x) + \sec^2 x - 1\right)\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tfrac{1}{2}\cos(2x) + \sec^2 x - \tfrac{1}{2}\right)\,dx$$

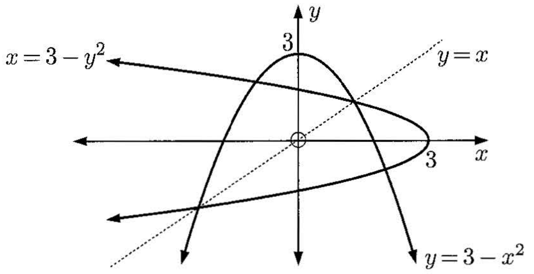
$$= \left[\tfrac{1}{4}\sin(2x) + \tan x - \tfrac{1}{2}x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \tfrac{1}{4}\sin\left(\tfrac{2\pi}{3}\right) + \tan\left(\tfrac{\pi}{3}\right) - \tfrac{\pi}{6} - \tfrac{1}{4}\sin\left(\tfrac{\pi}{3}\right) - \tan\left(\tfrac{\pi}{6}\right) + \tfrac{\pi}{12}$$

$$= \tfrac{1}{4}\left(\tfrac{\sqrt{3}}{2}\right) + \sqrt{3} - \tfrac{\pi}{6} - \tfrac{1}{4}\left(\tfrac{\sqrt{3}}{2}\right) - \tfrac{1}{\sqrt{3}} + \tfrac{\pi}{12}$$

$$= \sqrt{3} - \tfrac{1}{\sqrt{3}} - \tfrac{\pi}{12}$$

4 **a**

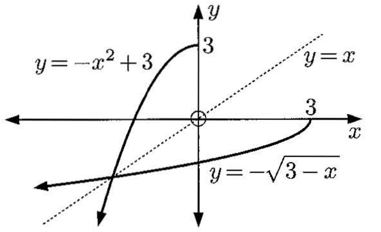


b The reflection $x = 3 - y^2$ is not a function since, for some values of x, there are two corresponding values of y. For example, when $x = 0$, $y = -\sqrt{3}$ or $\sqrt{3}$. Therefore, the reflection is not the inverse of $y = f(x)$.

c For the function $y = -x^2 + 3$, $x \leqslant 0$, the inverse function is $x = -y^2 + 3$, $y \leqslant 0$

$$\therefore \quad y^2 = 3 - x$$

$$\therefore \quad y = -\sqrt{3-x} \qquad \{\text{since } y \leqslant 0\}$$



5 $f(x) = 2x^3 - x^2 - 8x - 5$

$f(-1) = 0$ hence $(x+1)$ is a factor.

$\therefore \quad 2x^3 - x^2 - 8x - 5 = (x+1)(2x^2 + ax - 5)$

Equating coefficients of x^2, $\quad -1 = 2 + a$

$$\therefore \quad a = -3$$

So, $\quad 2x^3 - x^2 - 8x - 5$

$$= (x+1)(2x^2 - 3x - 5)$$

$$= (x+1)(x+1)(2x-5)$$

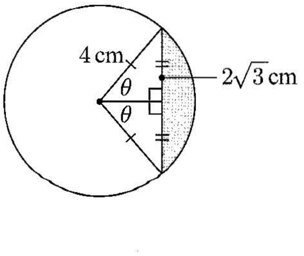
$$= (x+1)^2(2x-5)$$

$f(x) > 0$ if $2x - 5 > 0$, $x \neq -1$

$\{\text{since } (x+1)^2 > 0 \text{ for all } x \neq -1\}$

$\therefore \quad f(x) > 0$ if $x > \frac{5}{2}$.

6



$$\sin\theta = \tfrac{2\sqrt{3}}{4} = \tfrac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = \tfrac{\pi}{3}$$

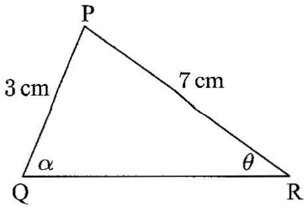
$$\text{Area} = \tfrac{1}{2}r^2 2\theta - \tfrac{1}{2}r^2\sin(2\theta)$$

$$= \tfrac{1}{2} \times r^2\left(\tfrac{2\pi}{3} - \sin\tfrac{2\pi}{3}\right)$$

$$= 8\left(\tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\right)\ \text{cm}^2$$

$$= \left(\tfrac{16\pi}{3} - 4\sqrt{3}\right)\ \text{cm}^2$$

7



By the sine rule:

$$\frac{\sin\alpha}{7} = \frac{\sin\theta}{3}$$

$$\therefore \quad \frac{\sin\alpha}{7} = \frac{\sqrt{3}}{14}$$

$$\therefore \quad \sin\alpha = \tfrac{\sqrt{3}}{14} \times 7 = \tfrac{\sqrt{3}}{2}$$

$$\therefore \quad \alpha = 60^\circ \text{ or } 120^\circ$$

8

$$3\cos 2\theta + 2 = 7\sin\theta$$

$$\therefore \quad 3(1 - 2\sin^2\theta) + 2 - 7\sin\theta = 0$$

$$\therefore \quad 3 - 6\sin^2\theta + 2 - 7\sin\theta = 0$$

$$\therefore \quad 6\sin^2\theta + 7\sin\theta - 5 = 0$$

$$\therefore \quad (2\sin\theta - 1)(3\sin\theta + 5) = 0$$

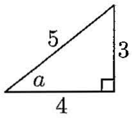
$$\therefore \quad \sin\theta = \tfrac{1}{2} \text{ or } -\tfrac{5}{3}$$

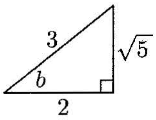
But $-1 < \sin\theta < 1$ for all θ.

$\therefore \quad \sin\theta = \frac{1}{2}$

9 Let $\arcsin\left(\frac{3}{5}\right) = a$ and $\arccos\left(\frac{2}{3}\right) = b$

$\therefore \quad \sin a = \frac{3}{5}$ and $\cos b = \frac{2}{3}$





$\therefore \quad \cos a = \frac{4}{5}$ and $\sin b = \frac{\sqrt{5}}{3}$

Now $\quad \cos\left(\arcsin\frac{3}{5} + \arccos\frac{2}{3}\right)$

$$= \cos(a + b)$$

$$= \cos a \cos b - \sin a \sin b$$

$$= \tfrac{4}{5} \times \tfrac{2}{3} - \tfrac{3}{5} \times \tfrac{\sqrt{5}}{3}$$

$$= \frac{8 - 3\sqrt{5}}{15}$$

10 $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{OY}$ if $\overrightarrow{OX} \bullet \overrightarrow{OY} = 0$

$$\therefore \quad \begin{pmatrix} 3\mu \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} \mu \\ 1 - 3\mu \\ 2\mu - 1 \end{pmatrix} = 0$$

$$\therefore \quad 3\mu^2 + 2(1 - 3\mu) + (2\mu - 1) = 0$$

$$\therefore \quad 3\mu^2 - 4\mu + 1 = 0$$

$$\therefore \quad (3\mu - 1)(\mu - 1) = 0$$

$$\therefore \quad \mu = \tfrac{1}{3} \text{ or } \mu = 1$$

11 Let $u = x + 2$

$\therefore \quad \frac{du}{dx} = 1$

and $x = u - 2$

$$\int \frac{x^2}{\sqrt{x+2}}\, dx$$

$$= \int \frac{(u-2)^2}{\sqrt{u}} \frac{du}{dx}\, dx$$

$$= \int \frac{u^2 - 4u + 4}{u^{\frac{1}{2}}}\, du$$

$$= \int u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}\, du$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \tfrac{2}{5}(x+2)^{\frac{5}{2}} - \tfrac{8}{3}(x+2)^{\frac{3}{2}} + 8\sqrt{x+2} + c$$

12 Let $P(x) = x^n + ax^2 - 6$.

Using the remainder theorem,

$P(1) = -3 \;\Rightarrow\; 1 + a - 6 = -3$

$\therefore \quad a = 2$

Also $\quad P(-3) = -15$

$$\therefore \quad (-3)^n + 2(-3)^2 - 6 = -15$$

$$\therefore \quad (-3)^n + 2 \times 9 - 6 = -15$$

$$\therefore \quad (-3)^n = -27 \quad \text{and so} \quad n = 3$$

$$\therefore \quad P(x) = x^3 + 2x^2 - 6$$

13 **a** $f : x \mapsto a + \dfrac{b}{x + c}$

f has a vertical asymptote $x = -2$, so $c = 2$.

f also has a horizontal asymptote $y = 3$. As x becomes large, f tends to 3. So, $a = 3$.

Since the graph passes through $(2, 4)$, $4 = a + \dfrac{b}{2 + c}$

$$\therefore \quad 4 = 3 + \frac{b}{4}$$

$$\therefore \quad b = 4$$

So, $f : x \mapsto 3 + \dfrac{4}{x+2}$ and $a = 3$, $b = 4$, $c - 2$.

b Domain of f is $x \in \mathbb{R}$, $x \neq -2$.

Range of f is $y \in \mathbb{R}$, $y \neq 3$.

c f is $y = 3 + \dfrac{4}{x+2}$ so f^{-1} is $x = 3 + \dfrac{4}{y+2}$

$$\therefore \quad x - 3 = \frac{4}{y+2}$$

$$\therefore \quad y + 2 = \frac{4}{x-3}$$

$$\therefore \quad f^{-1}(x) = \frac{4}{x-3} - 2$$

Domain of f^{-1} is $x \in \mathbb{R}$, $x \neq 3$.

Range of f^{-1} is $y \in \mathbb{R}$, $y \neq -2$.

14

$$\int_1^k 3\sqrt{10 - x}\, dx = 38$$

$$\therefore \quad \int_1^k (10 - x)^{\frac{1}{2}}\, dx = \tfrac{38}{3}$$

$$\therefore \quad \left[\tfrac{1}{-1}\frac{(10-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^k = \tfrac{38}{3}$$

$$\therefore \quad -\tfrac{2}{3}\left((10-k)^{\frac{3}{2}} - 9^{\frac{3}{2}}\right) = \tfrac{38}{3}$$

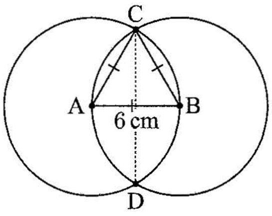
$$\therefore \quad (10 - k)^{\frac{3}{2}} - 27 = -19$$

$$\therefore \quad (10 - k)^{\frac{3}{2}} = 8$$

$$\therefore \quad 10 - k = 8^{\frac{2}{3}} = 4$$

$$\therefore \quad k = 6$$

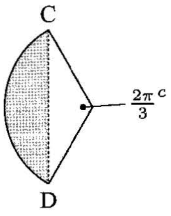
15



$\triangle$ABC and ABD are equilateral

$\therefore \quad \widehat{CBD} = 120^\circ = \frac{2\pi}{3}^{c}$

Thus, total area $= 2 \times$ shaded area



Area of segment $= \frac{1}{2}(r^2\theta - r^2\sin\theta)$

$\therefore \quad$ required area $= r^2\theta - r^2\sin\theta$

$$= 6^2\left(\tfrac{2\pi}{3} - \sin\left(\tfrac{2\pi}{3}\right)\right)$$

$$= 36\left(\tfrac{2\pi}{3} - \left(\tfrac{\sqrt{3}}{2}\right)\right)$$

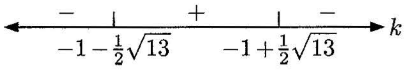
$$= 24\pi - 18\sqrt{3} \text{ cm}^2$$

16 $y = kx^2 - 3x + (k+2)$ cuts the x-axis in two distinct points if $\Delta > 0$.

$$\begin{aligned}\text{Now}\quad \Delta &= 9 - 4k(k+2)\\ &= 9 - 4k^2 - 8k\\ &= -\left(4k^2 + 8k - 9\right)\end{aligned}$$

$$\text{So,}\ \ \Delta = 0\ \ \text{if}\ \ k = \frac{-8 \pm \sqrt{64 - 4(4)(-9)}}{8} = \frac{-8 \pm \sqrt{208}}{8} = -1 \pm \tfrac{1}{2}\sqrt{13}$$

Thus Δ has sign diagram:



So, it cuts the x-axis in two distinct points if
$-1 - \frac{1}{2}\sqrt{13} < k < -1 + \frac{1}{2}\sqrt{13}$, $\ k \neq 0$.

17 Since $1 - 2i$ is a zero of $2z^3 - 9z^2 + 20z - 25$, $1 + 2i$ is also a zero.
These two zeros have a sum of 2 and a product of $1 + 4 = 5$ and so come from the quadratic factor $z^2 - 2z + 5$.
Thus, $2z^3 - 9z^2 + 20z - 25 = (z^2 - 2z + 5)(2z - 5)$
$\therefore$ the other zeros are $z = \frac{5}{2}$ and $z = 1 + 2i$.

18 Writing the system of equations in augmented matrix form

$$\left[\begin{array}{ccc|c} 2 & -2 & k & 0\\ 1 & 0 & 4 & 0\\ k & 1 & 1 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 2 & -2 & k & 0\\ k & 1 & 1 & 0\end{array}\right] \begin{array}{l} R_1 \to R_2\\ R_2 \to R_1\\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 0 & -2 & k-8 & 0\\ 0 & 1 & 1-4k & 0\end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - kR_1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 0 & 1 & 1-4k & 0\\ 0 & -2 & k-8 & 0\end{array}\right] \begin{array}{l} \\ R_2 \to R_3\\ R_3 \to R_2\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 0 & 1 & 1-4k & 0\\ 0 & 0 & -7k-6 & 0\end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 + 2R_2\end{array}$$

and this has a non-zero solution if
$-7k - 6 = 0$ or $k = -\frac{6}{7}$.

19 **a** $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\therefore\ \sin(A+B) + \sin(A-B) = 2\sin A\cos B$

b
$$\begin{aligned} f(x) &= \sin 5x\cos 2x\\ &= \tfrac{1}{2}(2\sin 5x\cos 2x)\\ &= \tfrac{1}{2}(\sin 7x + \sin 3x)\\ &= \underset{\substack{\uparrow\\ \text{period } \frac{2\pi}{7} = \frac{6\pi}{21}}}{\tfrac{1}{2}\sin 7x} + \underset{\substack{\uparrow\\ \text{period } \frac{2\pi}{3} = \frac{14\pi}{21}}}{\tfrac{1}{2}\sin 3x}\end{aligned}$$

$\sin 7x$ repeats after $\frac{6\pi}{21}, \frac{12\pi}{21}, \frac{18\pi}{21}, \frac{24\pi}{21}, \ldots$

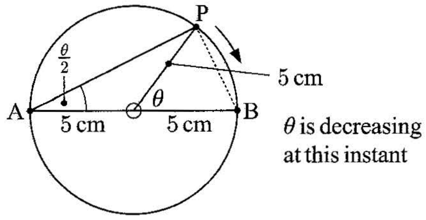
$\sin 3x$ repeats after $\frac{14\pi}{21}, \frac{28\pi}{21}, \frac{42\pi}{21}, \ldots$

LCM of 6 and 14 is 42 $\ \therefore\ $ period $= \frac{42\pi}{21} = 2\pi$

c
$$\begin{aligned} &\int \sin 5x\cos 2x\,dx\\ &= \int \left(\tfrac{1}{2}\sin 7x + \tfrac{1}{2}\sin 3x\right)dx\\ &= \tfrac{1}{2}\left(\tfrac{1}{7}\right)(-\cos 7x) + \tfrac{1}{2}\left(\tfrac{1}{3}\right)(-\cos 3x) + c\\ &= -\tfrac{1}{14}\cos 7x - \tfrac{1}{6}\cos 3x + c\end{aligned}$$

d
$$\begin{aligned} &\int_0^{\frac{\pi}{3}} \sin 5x\cos 2x\,dx\\ &= \left[-\tfrac{1}{14}\cos 7x - \tfrac{1}{6}\cos 3x\right]_0^{\frac{\pi}{3}}\\ &= -\tfrac{1}{14}\cos\left(\tfrac{7\pi}{3}\right) - \tfrac{1}{6}\cos\pi + \tfrac{1}{14}\cos 0 + \tfrac{1}{6}\cos 0\\ &= -\tfrac{1}{28} + \tfrac{1}{6} + \tfrac{1}{14} + \tfrac{1}{6}\\ &= \tfrac{31}{84}\end{aligned}$$

20



$$\frac{d\theta}{dt} = \frac{-1\text{ rev}}{10\pi\text{ s}} = \frac{-2\pi^c}{10\pi\text{ s}} = -\tfrac{1}{5}{}^c\,\text{s}^{-1}$$

$\text{P}\widehat{\text{A}}\text{B} = \frac{\theta}{2}$ {angle at centre theorem}

$$\begin{aligned} A &= \tfrac{1}{2}(\text{AP})(\text{PB})\\ \therefore\ A &= \tfrac{1}{2}\left[10\cos\left(\tfrac{\theta}{2}\right)\right]\left[10\sin\left(\tfrac{\theta}{2}\right)\right]\\ \therefore\ A &= 50\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\\ \therefore\ A &= 25\sin\theta\end{aligned}$$

$$\therefore\ \frac{dA}{dt} = 25\cos\theta\,\frac{d\theta}{dt}$$

Particular case:
When $\frac{\theta}{2} = \frac{\pi}{3}$, $\ \theta = \frac{2\pi}{3}$

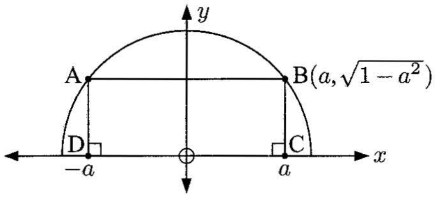
$$\therefore\ \frac{dA}{dt} = 25\cos\left(\tfrac{2\pi}{3}\right)\left(-\tfrac{1}{5}\right) = -5\left(-\tfrac{1}{2}\right) = 2.5$$

$\therefore$ at this instant the area is increasing at 2.5 cm^2 per second.

21 Let $x = \sin u$, $\dfrac{dx}{du} = \cos u$

$$\begin{aligned}\therefore\ \int_0^{\frac{1}{2}} \sqrt{1 - x^2}\,dx &= \int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 u}\,\cos u\,du\\ &= \int_0^{\frac{\pi}{6}} \cos^2 u\,du\\ &= \int_0^{\frac{\pi}{6}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2u\right)du\\ &= \left[\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u\right]_0^{\frac{\pi}{6}}\\ &= \tfrac{\pi}{12} + \tfrac{1}{4}\sin\left(\tfrac{\pi}{3}\right) - 0 - 0\\ &= \tfrac{\pi}{12} + \tfrac{1}{4}\tfrac{\sqrt{3}}{2}\\ &= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{8}\end{aligned}$$

22

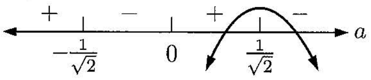


a Area, $A = 2a\sqrt{1 - a^2}$ $\qquad \therefore\ A^2 = 4a^2(1 - a^2)$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad = 4a^2 - 4a^4$

b $2A\dfrac{dA}{da} = 8a - 16a^3 \Rightarrow \dfrac{dA}{dt} = \dfrac{8a(1 - 2a^2)}{2A}$

$$\therefore\ \frac{dA}{dt} = \frac{4a(1 + \sqrt{2}a)(1 - \sqrt{2}a)}{A}$$

$\frac{dA}{dt}$ has sign diagram:

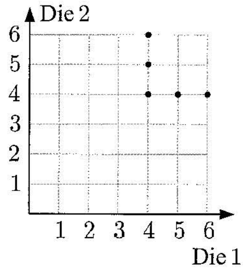


But $a > 0$, so we consider the local maximum at $a = \frac{1}{\sqrt{2}}$.

$\therefore \quad a = \frac{1}{\sqrt{2}}$ when the area is a maximum.

c $A_{\max} = 2 \times \frac{1}{\sqrt{2}} \times \sqrt{(1-\frac{1}{2})} = \frac{2}{\sqrt{2}}\sqrt{(\frac{1}{2})} = 1 \text{ unit}^2$

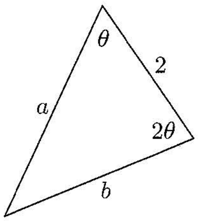
23 The possible outcomes when two dice are rolled are shown on the grid:



There are 5 outcomes where a 4 is scored.

Assuming fair dice are used, the probability of this occurence is $\frac{5}{36}$.

24 By the sine rule,



$$\frac{\sin 2\theta}{a} = \frac{\sin\theta}{b}$$

$$\therefore \quad \frac{2\sin\theta\cos\theta}{a} = \frac{\sin\theta}{b}$$

$$\therefore \quad \cos\theta = \frac{a}{2b} \quad .... \ (1)$$

By the cosine rule,

$$b^2 = a^2 + 2^2 - 2(a)(2)\cos\theta$$

$$\therefore \quad b^2 = a^2 + 4 - 4a\left(\frac{a}{2b}\right) \quad \{\text{using (1)}\}$$

$$\therefore \quad b^2 = a^2 + 4 - \frac{2a^2}{b}$$

$$\therefore \quad b^3 = a^2b + 4b - 2a^2$$

$$\therefore \quad b^3 - 4b = a^2(b-2)$$

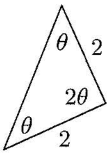
$$\therefore \quad b(b^2-4) = a^2(b-2)$$

$$\therefore \quad b(b+2)(b-2) = a^2(b-2)$$

$$\therefore \quad b(b+2) = a^2 \quad \{\text{p.v. } b \neq 2\}$$

$$\therefore \quad a = \sqrt{b(b+2)}$$

In the special case when $b = 2$ the triangle is isosceles and



$\theta + \theta + 2\theta = 180°$

$\therefore \quad \theta = 45°$ and $2\theta = 90°$

$\therefore$ the triangle is right angled.

In this case, $a^2 = 2^2 + 2^2 = 8 = 2 \times (2+2)$

So, again we find $a = \sqrt{b(b+2)}$.

25

$$\int_0^{\frac{\pi}{4}} \tan^n x\,dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$$

$$= \int_0^{\frac{\pi}{4}} [\tan x]^{n-2}(\tan^2 x + 1)\,dx$$

$$= \int_0^{\frac{\pi}{4}} [\tan x]^{n-2}\sec^2 x\,dx$$

$$= \left[\frac{[\tan x]^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n-1}\left((\tan\tfrac{\pi}{4})^{n-1} - (\tan 0)^{n-1}\right)$$

$$= \frac{1}{n-1}(1-0)$$

$$= \frac{1}{n-1}$$

SOLUTIONS TO EXAMINATION PRACTICE SET 2

1 **a** $\frac{1}{2} < r < 1$ so the series converges.

$$u_2 = u_1 r = 6 \quad \text{and} \quad S = 49$$

$$\therefore \quad \frac{u_1}{1-r} = 49$$

$$\therefore \quad \frac{6}{r} = 49(1-r)$$

$$\therefore \quad 6 = 49r - 49r^2$$

$$\therefore \quad 49r^2 - 49r + 6 = 0$$

$$\therefore \quad (7r-1)(7r-6) = 0$$

$$\therefore \quad r = \tfrac{1}{7} \text{ or } \tfrac{6}{7}$$

But $\frac{1}{2} < r < 1$, so $r = \frac{6}{7}$

b $u_1 r = 6$ so $u_1\left(\frac{6}{7}\right) = 6$ and hence $u_1 = 7$.

So, $u_n = u_1 r^{n-1} = 7\left(\frac{6}{7}\right)^{n-1}$.

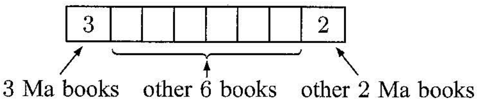
2 **a** With no restrictions, there are $8! = 40\,320$ different orderings.

b The mathematics books can be in one block in 3! ways.

This block plus the other 5 blocks can be ordered in 6! ways.

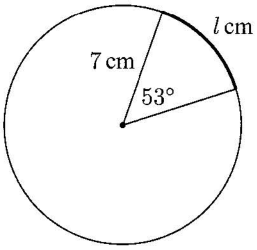
$\therefore$ total number is $3!6! = 4320$ ways.

c



$\therefore$ there are $3 \times 2 \times 6! = 4320$ ways.

3



a Perimeter

$= 2r + l$

$= 2 \times 7 + 7\left(53 \times \frac{\pi}{180}\right)$

≈ 20.5 cm

b Area

$= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 7^2 \times \left(53 \times \frac{\pi}{180}\right)$

$\approx 22.7 \text{ cm}^2$

4 For $\left(2x^2 + \frac{1}{x}\right)^9$, $T_{r+1} = \binom{9}{r}(2x^2)^{9-r}\left(\frac{1}{x}\right)^r$

$$= \binom{9}{r}2^{9-r}x^{18-2r}x^{-r}$$

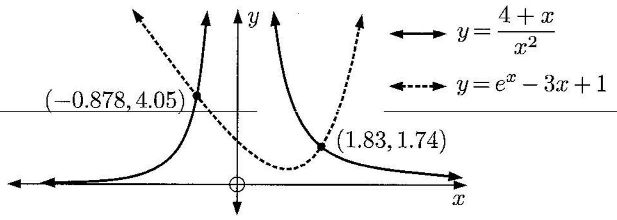
$$= \binom{9}{r}2^{9-r}x^{18-3r}$$

So, if we let $r = 6$, $T_7 = \binom{9}{6}2^3x^0$

So, the constant term is $\binom{9}{6} \times 8 = 672$.

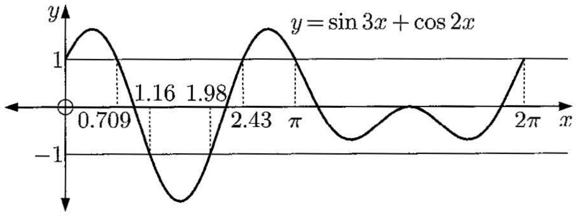
5 $x^2y = 4 + x \ \Rightarrow \ y = \frac{4+x}{x^2}$ providing $x \neq 0$

The graph of $y = \frac{4+x}{x^2}$ and $y = e^x - 3x + 1$ is shown.



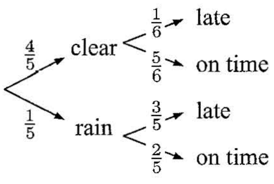
6 arccos$(\sin 3x + \cos 2x)$ is defined where $-1 \leqslant \sin 3x + \cos 2x \leqslant 1$.

The graph of $y = \sin 3x + \cos 2x$ is obtained using technology:



$\therefore \quad x = 0, \quad 0.709 \leqslant x \leqslant 1.16, \quad 1.98 \leqslant x \leqslant 2.43,$
$\pi \leqslant x \leqslant 2\pi$

7



$$\begin{aligned} &\text{P(Raining} \mid \text{Late)} \\ &= \frac{\text{P(Raining} \cap \text{Late)}}{\text{P(Late)}} \\ &= \frac{\frac{1}{5} \times \frac{3}{5}}{\frac{4}{5} \times \frac{1}{6} + \frac{1}{5} \times \frac{3}{5}} \\ &= \tfrac{9}{19} \end{aligned}$$

8 $X \sim \text{B}(7, p)$

If $\text{P}(X = 4) = 0.25$ then $\binom{7}{4} p^4(1-p)^3 = 0.25$

Using technology $p \approx 0.464$ or 0.674

9 Since the vectors are perpendicular

$(\lambda\mathbf{i} + \mathbf{j} - \lambda\mathbf{k}) \bullet (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 0$

$\therefore \quad 3\lambda - 4 - \lambda = 0 \quad \text{or} \quad \lambda = 2$

The vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ has length $\sqrt{2^2 + 1^2 + 2^2} = 3$

A vector of unit length parallel to $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

10 We assume the number of students in the school is large enough to use the Binomial distribution.

If X is the number of students who travel by bike, then

$X \sim \text{B}(7, \frac{2}{7})$ and $\text{P}(X = 4) = \binom{7}{4}(\frac{2}{7})^4(\frac{5}{7})^3$
≈ 0.0850

11 Let X be the number of fish not suitable for sale in 20 chosen from the box. As these are selected from a box of 1000 (a large number), we assume that $X \sim \text{B}(20, 0.037)$.

a If all are suitable for sale, then $X = 0$.

$\text{P}(X = 0) = \binom{20}{0}(0.037)^0(0.963)^{20}$
≈ 0.470

b $\text{P}(X = 1) = \binom{20}{1}(0.037)(0.963)^{19}$
≈ 0.362

12 For $(ax+3)^5$, $T_{r+1} = \binom{5}{r}(ax)^{5-r}3^r = \binom{5}{r}a^{5-r}3^r x^{5-r}$

$\therefore$ the coefficient of x^4 is $\binom{5}{1}a^4 3^1$.

For $(ax+3)^7$, $T_{r+1} = \binom{7}{r}(ax)^{7-r}3^r = \binom{7}{r}a^{7-r}3^r x^{7-r}$

$\therefore$ the coefficient of x^5 is $\binom{7}{2}a^5 3^2$.

Thus $\binom{5}{1}a^4 3^1 = \binom{7}{2}a^5 3^2$ and so $15a^4 = 21 \times 9a^5$

$\therefore$ as $a \neq 0$, $a = \dfrac{15}{21 \times 9} = \frac{5}{63}$.

13 **a** $(g \circ f)(4) = g(f(4))$
$= g\left(\frac{1}{4+5}\right)$
$= g\left(\frac{1}{9}\right)$
$= 3\left(\frac{1}{9}\right)$
$= \frac{1}{3}$

b g is $y = 3x$
$\therefore \quad g^{-1}$ is given by $x = 3y$
$\therefore \quad g^{-1}(x) = \dfrac{x}{3}$

c Domain of g^{-1} is $x \in \mathbb{R}$.

14 $f(x) = ax^3 + bx^2 + cx + d$

$\therefore \quad f'(x) = 3ax^2 + 2bx + c$

Now $f(0) = 1, \qquad f'(0) = 0$
$f(-2) = -2, \qquad f'(-2) = 0$

Using $f(0) = 1$ we have $d = 1$.

Using $f'(0) = 0$ we have $3a(0) + 2b(0) + c = 0$.

So, $c = 0$, $d = 1$

$f(-2) = -2 \quad \therefore \quad a(-8) + b(4) + 0(-2) + 1 = -2$
$\therefore \quad -8a + 4b + 1 = -2$
$\therefore \quad -8a + 4b = -3 \quad .... \ (1)$

$f'(-2) = 0 \quad \therefore \quad 3a(4) + 2b(-2) + 0 = 0$
$\therefore \quad 12a - 4b = 0$
$\therefore \quad b = 3a \quad .... \ (2)$

Substituting (2) into (1), $-8a + 12a = -3$
$\therefore \quad 4a = -3$
$\therefore \quad a = -\frac{3}{4}$ and so $b = -\frac{9}{4}$

Thus $a = -\frac{3}{4}$, $b = -\frac{9}{4}$, $c = 0$, $d = 1$.

15 **a** Let X be the measured speed of a car travelling at 60 km h^{-1}.

The error in reading the speed is $E = X - 60$.

So, $X = 60 + E$.

b Since $E \sim \text{N}(0, \sigma^2) \qquad X \sim \text{N}(60, \sigma^2)$

$\text{P}(X \geqslant 65) = 0.01$
$\therefore \quad \text{P}(X \leqslant 65) = 0.99$
$\therefore \quad \text{P}\left(\dfrac{X - 60}{\sigma} \leqslant \dfrac{65 - 60}{\sigma}\right) = 0.99$
$\therefore \quad \text{P}\left(Z \leqslant \dfrac{5}{\sigma}\right) = 0.99$
$\therefore \quad \dfrac{5}{\sigma} \approx 2.33$
$\therefore \quad \sigma \approx 2.15$

16 Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$\therefore \quad \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$ {since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$}

$\therefore \quad \mathbf{a} \times \mathbf{b} = -(\mathbf{a} \times \mathbf{c}) \quad .... \ (1)$

Also $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times \mathbf{c} = \mathbf{0}$

$\therefore \quad \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$ {since $\mathbf{c} \times \mathbf{c} = \mathbf{0}$}

$\therefore \quad \mathbf{b} \times \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \quad .... \ (2)$

From (1) and (2), $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

17 $\displaystyle\int_0^p \left(x^3 + x\right) dx = \frac{15}{4}, \quad p > 0$

$\therefore \quad \left[\dfrac{x^4}{4} + \dfrac{x^2}{2}\right]_0^p = \frac{15}{4}$

$\therefore \quad \dfrac{p^4}{4} + \dfrac{p^2}{2} - 0 = \frac{15}{4}$

$\therefore \quad p^4 + 2p^2 = 15$

$\therefore \quad p^4 + 2p^2 - 15 = 0$

$\therefore \quad (p^2 + 5)(p^2 - 3) = 0$

$\therefore \quad p^2 = -5$ or 3

But p is real and positive, so $p = \sqrt{3}$.

18 $(f \circ g)(x) = f(g(x)) = f(3x - 2) = x + 2$ {given}

and $f(9x - 8) = f(3(3x - 2) - 2)$
$= f(3y - 2)$ {letting $y = 3x - 2$}
$= y + 2$
$= 3x$

19 Let X be the diameter of a disc, then $X \sim \text{N}(73,\ 1.1^2)$
Hence, $\text{P}(X > 75) \approx 0.0345$.

20 $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$
Since A and B are independent $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$
Hence, letting $\text{P}(B) = p$, $0.75 = 0.35 + p - 0.35p$
So, $0.65p = 0.40$
$\therefore\ p = \frac{40}{65} = \frac{8}{13}$

21 Let X be the number of blonde children. Then $X \sim \text{B}(5,\ \frac{1}{7})$.

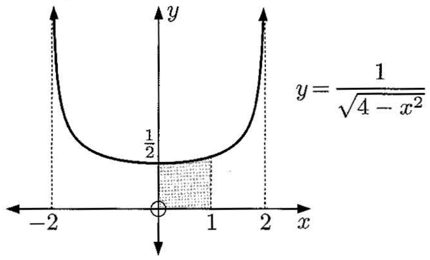
a $\text{E}(X) = np = 5 \times \frac{1}{7} = \frac{5}{7}$

Expected number of blonde children is $\frac{5}{7}$.

b
$$\begin{aligned}&\text{P}(X = 3)\\ &= \binom{5}{3}\left(\tfrac{1}{7}\right)^3\left(\tfrac{6}{7}\right)^2\\ &\approx 0.0214\end{aligned}$$

c
$$\begin{aligned}&\text{P}(X > 3)\\ &= 1 - \text{P}(X \leqslant 3)\\ &\approx 1 - 0.998\,16\\ &\approx 0.001\,84\end{aligned}$$

22



a
$$\begin{aligned}\text{Area} = \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx &= \left[\arcsin\left(\frac{x}{2}\right)\right]_0^1\\ &= \arcsin\left(\tfrac{1}{2}\right) - \arcsin(0)\\ &= \tfrac{\pi}{6} - 0\\ &= \tfrac{\pi}{6}\ \text{units}^2\end{aligned}$$

b
$$\text{Volume} = \pi \int_0^1 \frac{1}{4-x^2}\,dx$$
$$\approx 0.863\ \text{units}^3\ \ \{\text{using technology}\}$$

For the exact value

let $\dfrac{1}{4-x^2} = \dfrac{A}{2+x} + \dfrac{B}{2-x}$

$\therefore\ A(2-x) + B(2+x) = 1$

If $x = 2$, $4B = 1$ $\therefore\ B = \frac{1}{4}$

If $x = -2$, $4A = 1$ $\therefore\ A = \frac{1}{4}$

$$\begin{aligned}\therefore\ \text{volume} = \pi \int_0^1 \left(\frac{\frac{1}{4}}{x+2} + \frac{\frac{1}{4}}{2-x}\right) dx &= \tfrac{1}{4}\pi\left[\ln|x+2| - \ln|2-x|\right]_0^1\\ &= \tfrac{\pi}{4}(\ln 3 - \ln 1 - \ln 2 + \ln 2)\\ &= \frac{\pi \ln 3}{4}\ \text{units}^3\end{aligned}$$

23

a
$$\begin{aligned}\tfrac{2}{7} + \tfrac{1}{7} + \tfrac{3}{14} + \tfrac{1}{14} + \tfrac{1}{7} + y &= 1\\ \therefore\ \tfrac{6}{7} + y &= 1\\ \therefore\ y &= \tfrac{1}{7}\end{aligned}$$

b
$$\begin{aligned}&\text{E}(X)\\ &= 1 \times \tfrac{2}{7} + 2 \times \tfrac{1}{7} + 3 \times \tfrac{3}{14} + 4 \times \tfrac{1}{14} + 5 \times \tfrac{1}{7} + 6 \times \tfrac{1}{7}\\ &= \tfrac{1}{14}(4 + 4 + 9 + 4 + 10 + 12)\\ &= \tfrac{43}{14}\end{aligned}$$

$$\begin{aligned}&\text{Var}(X)\\ &= \text{E}(X^2) - (\text{E}(X))^2\\ &= 1^2(\tfrac{2}{7}) + 2^2(\tfrac{1}{7}) + 3^2(\tfrac{3}{14}) + 4^2(\tfrac{1}{14}) + 5^2(\tfrac{1}{7}) + 6^2(\tfrac{1}{7})\\ &\quad - \left(\tfrac{43}{14}\right)^2\\ &= \tfrac{629}{196}\end{aligned}$$

c If the die is tossed many times we expect the mean value of the tosses to be $\frac{43}{14}$ with standard deviation of $\sqrt{\frac{629}{196}}$.

24

a
$$\begin{aligned}&\cos(A+B) - \cos(A-B)\\ &= \cos A\cos B - \sin A \sin B\\ &\qquad - (\cos A \cos B + \sin A \sin B)\\ &= -2\sin A \sin B\end{aligned}$$

b P_n is "$\sin x + \sin 3x + \sin 5x + \ldots. + \sin(2n-1)x$
$= \dfrac{\sin^2 nx}{\sin x}$" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \sin x$

RHS $= \dfrac{\sin^2 x}{\sin x} = \sin x$

$\therefore$ P_1 is true.

(2) If P_k is true then

$\sin x + \sin 3x + \sin 5x + \ldots. + \sin(2k-1)x$
$= \dfrac{\sin^2 kx}{\sin x}$

$\therefore$ $\sin x + \sin 3x + \ldots. + \sin(2k-1)x$
$+ \sin(2k+1)x$

$$\begin{aligned}&= \frac{\sin^2 kx}{\sin x} + \sin(2k+1)x\\ &= \frac{\sin^2 kx + \sin(2k+1)x \sin x}{\sin x}\\ &= \frac{\sin^2 kx + (-\frac{1}{2})\left[\cos(2k+2)x - \cos 2kx\right]}{\sin x}\\ &= \frac{\sin^2 kx - \frac{1}{2}\cos 2(k+1)x + \frac{1}{2}\cos 2kx}{\sin x}\\ &= \frac{\frac{1}{2} - \frac{1}{2}\cos 2kx - \frac{1}{2}\cos 2(k+1)x + \frac{1}{2}\cos 2kx}{\sin x}\\ &= \frac{\frac{1}{2} - \frac{1}{2}\cos 2(k+1)x}{\sin x}\\ &= \frac{\sin^2(k+1)x}{\sin x} \qquad \{\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$.
{Principle of mathematical induction}

25

a $\overrightarrow{\text{AB}} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$, $\overrightarrow{\text{AC}} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$, and

$$\begin{aligned}\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & 0 & 1\\ -1 & -3 & 3\end{vmatrix}\\ &= (0+3)\mathbf{i} - (-6+1)\mathbf{j} + (6)\mathbf{k}\\ &= 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}\ \text{ or } \begin{pmatrix}3\\5\\6\end{pmatrix}\end{aligned}$$

b **i** By definition, A and B lie on the plane ABC.
For A(3, 0, 2), $2(3) - 1(0) + 4(2) = 14$
so A lies on the plane $2x - y + 4z = 14$.

Similarly for B(1, 0, 3), $2(1) - 1(0) + 4(3) = 14$, so B lies on the plane $2x - y + 4z = 14$.
So A and B lie on both planes. Thus A and B are points on the river.

ii $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ Since A(3, 0, 2) lies on the line, an equation is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-2\lambda \\ 0 \\ 2+\lambda \end{pmatrix}$$

iii $\overrightarrow{BC} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{BC} \bullet \overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$$= 0$$

So $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{AB}$.
Hence B is the closest point on the river to C.

iv The shortest distance is $|BC| = \sqrt{1^2 + (-3)^2 + 2^2}$
$= \sqrt{14}$ units

SOLUTIONS TO EXAMINATION PRACTICE SET 3

1 **a** f is $y = x^2 + 4x, \quad -\infty < x \leqslant -2$

so f^{-1} is $x = y^2 + 4y, \quad -\infty < y \leqslant -2$

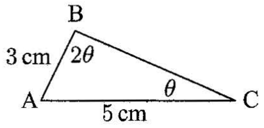
$\therefore \quad y^2 + 4y - x = 0$

$\therefore \quad y = \dfrac{-4 \pm \sqrt{16 + 4x}}{2} = -2 \pm \sqrt{4+x}$

But $y \leqslant -2$, so $y = -2 - \sqrt{4+x}$

b
$$\begin{aligned}(g \circ f)(-3) &= g\left(f(-3)\right) \\ &= g\left((-3)^2 + 4(-3)\right) \\ &= g(-3) \\ &= \sqrt{3 - 2(-3)} \\ &= \sqrt{3+6} = 3\end{aligned}$$

2



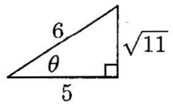
By the sine rule, $\dfrac{\sin 2\theta}{5} = \dfrac{\sin\theta}{3}$

$\therefore \quad 3\sin 2\theta = 5\sin\theta$

$\therefore \quad 6\sin\theta\cos\theta = 5\sin\theta$

$\therefore \quad \sin\theta(6\cos\theta - 5) = 0$

$\therefore$ as $\sin\theta \neq 0, \quad \cos\theta = \frac{5}{6}$



$\therefore \quad \sin\theta = \frac{\sqrt{11}}{6}$

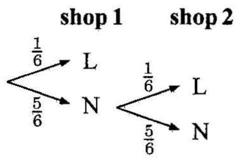
Let $\mathrm{B}\widehat{\mathrm{A}}\mathrm{C} = \alpha°$ then $\alpha + 3\theta = \pi$

$\therefore \quad \alpha = \pi - 3\theta$

$$\begin{aligned}\sin\alpha &= \sin(\pi - 3\theta) \\ &= \sin\pi\cos 3\theta - \cos\pi\sin 3\theta \\ &= (0)\cos 3\theta - (-1)\sin 3\theta \\ &= \sin 3\theta \\ &= \sin(2\theta + \theta) \\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\ &= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta \\ &= 2 \times \tfrac{\sqrt{11}}{6} \times \tfrac{25}{36} + \left(\tfrac{25}{36} - \tfrac{11}{36}\right) \times \tfrac{\sqrt{11}}{6} \\ &= \tfrac{8}{27}\sqrt{11}\end{aligned}$$

$\therefore \quad$ area $= \frac{1}{2} \times 3 \times 5 \times \frac{8}{27}\sqrt{11} = \frac{20}{9}\sqrt{11}$ units^2

3 A tree diagram shows the situation.



Probability she leaves it in shop 1 given it is missing is

$$= \frac{\text{P(shop 1 and missing)}}{\text{P(missing)}} = \frac{\frac{1}{6}}{\frac{5}{36} + \frac{1}{6}} = \frac{6}{11}$$

4

$$\cot 2A = \tfrac{3}{5}$$

$\therefore \quad \tan 2A = \frac{5}{3}$

$\therefore \quad \dfrac{2\tan A}{1 - \tan^2 A} = \frac{5}{3}$

$\therefore \quad 6\tan A = 5 - 5\tan^2 A$

$\therefore \quad 5\tan^2 A + 6\tan A - 5 = 0$

Thus $\tan A = \dfrac{-6 \pm \sqrt{36 - 4(5)(-5)}}{10}$

$= \dfrac{-6 \pm \sqrt{136}}{10}$

$= \dfrac{-3 \pm \sqrt{34}}{5}$

But A is obtuse, so $\tan A = \dfrac{-3 - \sqrt{34}}{5}$.

$\therefore \quad \cot A = \left(\dfrac{5}{-3 - \sqrt{34}}\right)\left(\dfrac{-3 + \sqrt{34}}{-3 + \sqrt{34}}\right)$

$= \dfrac{-15 + 5\sqrt{34}}{9 - 34}$

$= \dfrac{-15 + 5\sqrt{34}}{-25}$

$= \frac{3}{5} - \frac{1}{5}\sqrt{34}$

5 **a** $y = mx + 16$ will meet the parabola $y = x^2 + 25$ where $mx + 16 = x^2 + 25$

$\therefore \quad x^2 - mx + 9 = 0$

The line will be a tangent if $\Delta = 0$

$\therefore \quad m^2 - 4 \times 9 = 0$

$\therefore \quad m^2 = 36$

$\therefore \quad m = \pm 6$

b $f(x) = x^2 + (2-k)x + k^2$

$f(x) > 0$ for all x if $\Delta < 0$

Now $\Delta = (2-k)^2 - 4k^2$
$= 4 - 4k + k^2 - 4k^2$
$= -(3k^2 + 4k - 4)$
$= -(3k - 2)(k + 2)$

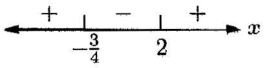
So, Δ has sign diagram:

So, $\Delta < 0$ if $k < -2$ or $k > \frac{2}{3}$.

6 $\log_3\left(4x^2 - 5x - 6\right) = 1 + 2\log_3 x$

For this to be a valid equation in real numbers, we must have $(4x+3)(x-2) > 0$ *and* $x > 0$.

Sign diagram of the quadratic is:



$\therefore \quad x < -\frac{3}{4}$ or $x > 2$. But $x > 0$, so $x > 2$.

$$\log_3(4x^2-5x-6) = 1 + 2\log_3(x)$$
$$\therefore\ \log_3(4x^2-5x-6) = \log_3(3) + \log_3(x^2)$$
$$\therefore\ \log_3(4x^2-5x-6) = \log_3(3x^2)$$
$$\therefore\ 4x^2-5x-6 = 3x^2$$
$$\therefore\ x^2-5x-6 = 0$$
$$\therefore\ (x+1)(x-6) = 0$$
$$\therefore\ x = -1 \text{ or } 6$$

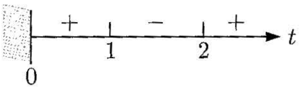
But $x > 2$ $\therefore$ $x = 6$ is the only solution.

7 $$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\arcsin x\right]_0^{\frac{1}{2}}$$
$$= \arcsin(\tfrac{1}{2}) - \arcsin(0)$$
$$= \tfrac{\pi}{6} - 0$$
$$= \tfrac{\pi}{6}$$

8 $$v = t^3 - 3t^2 + 2t$$
$$= t(t^2-3t+2)$$
$$= t(t-1)(t-2)$$

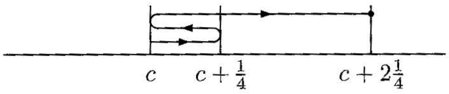
So, $v = 0$ at $t = 0, 1, 2$ s

Sign diagram of v:



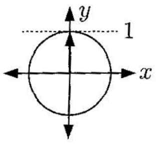
$\therefore$ reverses direction at $t = 1$, $t = 2$

$$s = \frac{t^4}{4} - \frac{3t^3}{3} + \frac{2t^2}{2} + c$$
$$\therefore\ s = \tfrac{1}{4}t^4 - t^3 + t^2 + c \text{ metres}$$
$$s(0) = c$$
$$s(1) = \tfrac{1}{4} - 1 + 1 + c = c + \tfrac{1}{4}$$
$$s(2) = 4 - 8 + 4 + c = c$$
$$s(3) = \tfrac{81}{4} - 27 + 9 + c = 2\tfrac{1}{4} + c$$



$$\text{Total distance travelled} = \tfrac{1}{4} + \tfrac{1}{4} + 2\tfrac{1}{4}$$
$$= 2\tfrac{3}{4} \text{ m or } 2.75 \text{ m}$$

9 $$\cos\theta + \sin\theta = \sqrt{2}, \quad 0 \leqslant \theta \leqslant 2\pi$$
$$\therefore\ (\cos\theta + \sin\theta)^2 = 2$$
$$\therefore\ \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = 2$$
$$\therefore\ 1 + \sin 2\theta = 2$$
$$\therefore\ \sin 2\theta = 1$$



But $0 \leqslant 2\theta \leqslant 4\pi$, so $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

$$\therefore\ \theta = \tfrac{\pi}{4}, \tfrac{5\pi}{4}$$

But $\theta = \frac{5\pi}{4}$ gives $\cos\theta + \sin\theta = -\sqrt{2}$

$$\therefore\ \theta = \tfrac{\pi}{4}$$

Hint: Squaring equations can produce incorrect solutions, so all solutions must be checked.

10 $$f(x) = e^{x\ln x}$$
$$\therefore\ f'(x) = e^{x\ln x}\left(1\ln x + x\left(\frac{1}{x}\right)\right)$$
$$= e^{x\ln x}(\ln x + 1)$$

Since $e^{x\ln x} > 0$ for all x, $f'(x)$ is zero only when $\ln x = -1$.

$\therefore$ $f'(x) = 0$ when $x = e^{-1}$

So, the x-coordinate of the stationary point is $\dfrac{1}{e}$.

11 $$S_3 = u_1\left(\frac{r^3-1}{r-1}\right) \quad \text{and} \quad S = \frac{u_1}{1-r}$$

So, $$\frac{u_1(r^3-1)}{r-1} = \tfrac{1}{2}u_1\left(\frac{1}{1-r}\right)$$
$$\therefore\ r^3 - 1 = -\tfrac{1}{2}$$
$$\therefore\ r^3 = \tfrac{1}{2}$$
$$\therefore\ r = \sqrt[3]{\tfrac{1}{2}}$$

12 $2 - i$ is a solution

$\therefore$ $2 + i$ is also a solution {theorem on real polynomials}

These have sum $= 4$ and product $= 5$ and so come from the factor $z^2 - 4z + 5$.

Thus $z^3 - 6z^2 + 13z - 10 = (z^2 - 4z + 5)(z - 2)$

$\therefore$ the other solutions are $2 + i$ and 2.

13 **a** Domain $= \{x \mid x \leqslant 5\}$. Range $= \{y \mid y \geqslant 0\}$.

b Domain $= \{x \mid x \in \mathbb{R}\}$. Range $= \{y \mid 0 \leqslant y \leqslant 0.5\}$.

14 **a** $$\int \frac{1+x}{4+x^2}\,dx$$
$$= \int \frac{1}{4+x^2}\,dx + \int \frac{x}{4+x^2}\,dx$$
$$= \tfrac{1}{2}\arctan\left(\frac{x}{2}\right) + \tfrac{1}{2}\int \frac{2x}{4+x^2}\,dx$$
$$= \tfrac{1}{2}\arctan\left(\frac{x}{2}\right) + \tfrac{1}{2}\ln\left|4+x^2\right| + c$$
$$= \tfrac{1}{2}\arctan\left(\frac{x}{2}\right) + \tfrac{1}{2}\ln\left(4+x^2\right) + c$$

{$4 + x^2 > 0$ for all x}

b We integrate by parts with $u = \ln x$ $\quad v' = x^2$

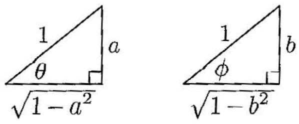
$$u' = \frac{1}{x} \qquad v = \frac{x^3}{3}$$

$$\therefore\ \int x^2\ln x\,dx = \frac{x^3\ln x}{3} - \int \frac{x^2}{3}\,dx$$
$$= \frac{x^3\ln x}{3} - \tfrac{1}{3}\frac{x^3}{3} + c$$
$$= \frac{x^3\ln x}{3} - \frac{x^3}{9} + c$$

15 **a** Consider $\sin(\arcsin a + \arcsin b)$

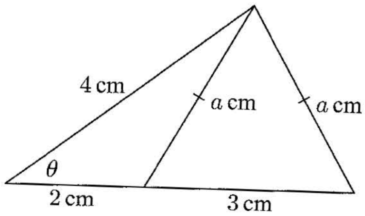
Let $\arcsin a = \theta$, $\arcsin b = \phi$

$\therefore$ $\sin\theta = a$ and $\sin\phi = b$



$$\sin(\arcsin a + \arcsin b) = \sin(\theta + \phi)$$
$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$= a\left(\sqrt{1-b^2}\right) + \left(\sqrt{1-a^2}\right)b$$
$$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$$

b



By the cosine rule,

$$\cos\theta = \frac{4^2+5^2-a^2}{2\times4\times5} = \frac{41-a^2}{40} \quad .... (1)$$

$$\text{and} \quad \cos\theta = \frac{4^2+2^2-a^2}{2\times4\times2} = \frac{20-a^2}{16} \quad .... (2)$$

$$\text{Thus} \quad \frac{41-a^2}{40} = \frac{20-a^2}{16}$$

$$\therefore\ 82-2a^2 = 100-5a^2$$

$$\therefore\ 3a^2 = 18$$

$$\therefore\ a^2 = 6$$

$$\therefore\ a = \sqrt{6} \quad \{\text{as } a>0\}$$

16 $$f(x) = \log_3\left(\frac{x^2+1}{3x+1}\right) = \frac{\ln\left(\dfrac{x^2+1}{3x+1}\right)}{\ln 3}$$

$$\therefore\ f(x) = \frac{1}{\ln 3}\left[\ln(x^2+1) - \ln(3x+1)\right]$$

$$\therefore\ f'(x) = \frac{1}{\ln 3}\left[\frac{2x}{x^2+1} - \frac{3}{3x+1}\right]$$

So, $f'(x)=0$ when $\dfrac{2x}{x^2+1} = \dfrac{3}{3x+1}$

$$\therefore\ 6x^2+2x = 3x^2+3$$

$$\therefore\ 3x^2+2x-3=0$$

$$\therefore\ x = \frac{-2\pm\sqrt{4-4(3)(-3)}}{6}$$

$$\therefore\ x = \frac{-2\pm\sqrt{40}}{6} = \frac{-1\pm\sqrt{10}}{3}$$

But for $f(x)$ to exist, $\dfrac{x^2+1}{3x+1}$ must be >0.

As $x^2+1>0$ for all x then $3x+1>0$ $\therefore$ $x>-\frac{1}{3}$

Consequently, $x = \dfrac{-1+\sqrt{10}}{3}$

So, only one stationary point exists and it is at $x = \dfrac{\sqrt{10}-1}{3}$.

17 **a** L_1 meets the plane $2x+y-z=2$

when $2(3\lambda-4)+(\lambda+2)-(2\lambda-1)=2$

$\therefore$ $5\lambda-5=2$ $\quad\therefore$ $\lambda=\frac{7}{5}$

$\therefore$ $x=\frac{1}{5}$, $y=\frac{17}{5}$, $z=\frac{9}{5}$

So, they meet at $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$.

b The lines meet if

$$(3\lambda-4) = \frac{(\lambda+2)-5}{2} = \frac{-(2\lambda-1)-1}{2}$$

From the first equation $2(3\lambda-4)=\lambda-3$

$$\therefore\ 6\lambda-8=\lambda-3$$

$$\therefore\ \lambda=1$$

Substituting $\lambda=1$ in the equation for L_2 we have

$$(3-4) = \frac{(1+2)-5}{2} = \frac{-(2-1)-1}{2}$$

$$\therefore\ -1 = \quad -1 \quad = -1$$

As this checks, the point $x=-1$, $y=3$, $z=1$ lies on both lines.

c L_1 has direction $\begin{pmatrix}3\\1\\2\end{pmatrix}$. L_2 has direction $\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

As $(-1, 3, 1)$ is a point on both lines, the equation of the plane that contains both L_1 and L_2 is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix} + s\begin{pmatrix}3\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$$or \quad \begin{pmatrix}3\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ 1 & 2 & -2\end{vmatrix}$$

$$= \begin{vmatrix}1 & 2\\ 2 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 2\\ 1 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}\mathbf{k}$$

$$= (-2-4)\mathbf{i} - (-6-2)\mathbf{j} + (6-1)\mathbf{k}$$

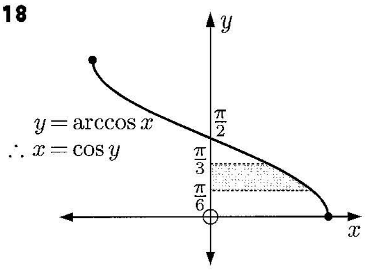
$$= -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$$

$\therefore$ the equation is

$$-6x+8y+5z = -6(-1)+8(3)+5(1)$$

which is $-6x+8y+5z=35$.

18



$$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x^2\,dy$$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 y\,dy$$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tfrac{1}{2}+\tfrac{1}{2}\cos 2y\,dy$$

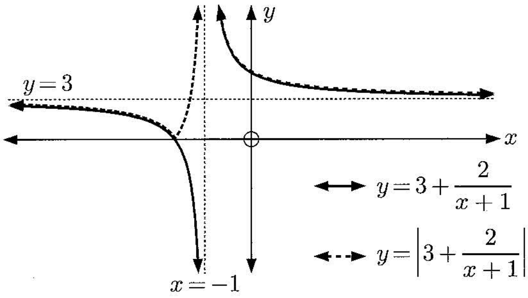
$$= \pi\left[\tfrac{1}{2}y+\tfrac{1}{4}\sin 2y\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \pi\left(\tfrac{\pi}{6}+\tfrac{1}{4}\sin\left(\tfrac{2\pi}{3}\right)\right.$$
$$\left.-\tfrac{\pi}{12}-\tfrac{1}{4}\sin\left(\tfrac{\pi}{3}\right)\right)$$

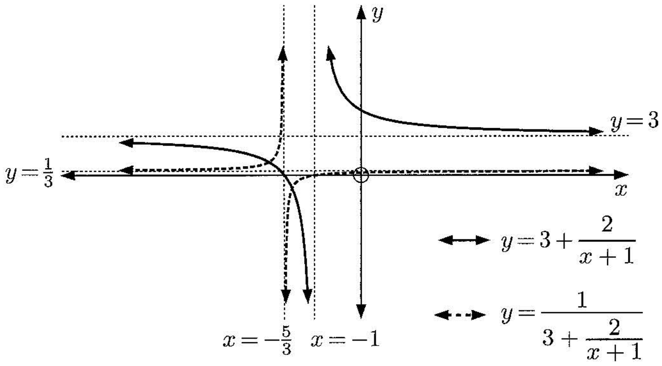
$$= \pi\left(\tfrac{\pi}{12}\right)$$

$$= \tfrac{\pi^2}{12}\ \text{units}^3$$

19 Vertical asymptote is $x=-1$.
Horizontal asymptote is $y=3$.



Note that the graphs of $3+\dfrac{2}{x+1}$ and $\left|3+\dfrac{2}{x+1}\right|$ coincide for all $y \geqslant 0$.



20 **a** $\cos x + \sin x\cos x + \sin^2 x\cos x + ...$ is geometric with $u_1 = \cos x$, $r = \sin x$.

Now as $0 < x < \frac{\pi}{2}$, $0 < \sin x < 1$,

So, the series converges and $S = \dfrac{u_1}{1-r} = \dfrac{\cos x}{1-\sin x}$.

b If $S=\sqrt{3}$, $\dfrac{\cos x}{1-\sin x}=\sqrt{3}$

$$\therefore \quad \frac{\cos^2 x}{1-2\sin x+\sin^2 x}=3$$

$$\therefore \quad \cos^2 x = 3-6\sin x+3\sin^2 x$$

$$\therefore \quad 3\sin^2 x-6\sin x+3=1-\sin^2 x$$

$$\therefore \quad 4\sin^2 x-6\sin x+2=0$$

$$\therefore \quad 2\sin^2 x-3\sin x+1=0$$

$$\therefore \quad (2\sin x-1)(\sin x-1)=0$$

$$\therefore \quad \sin x=\tfrac{1}{2} \text{ or } 1$$

But $0<\sin x<1 \quad \{0<x<\frac{\pi}{2}\}$

So, $\sin x=\frac{1}{2}$ and so $x=\frac{\pi}{6}$

21 **a** $\text{LHS}=\dfrac{\cos x}{1-\sin x}$

$$=\left(\frac{\cos x}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)$$

$$=\frac{\cos x+\sin x\cos x}{1-\sin^2 x}$$

$$=\frac{\cos x+\sin x\cos x}{\cos^2 x}$$

$$=\frac{1}{\cos x}+\frac{\sin x}{\cos x}$$

$=\sec x+\tan x$ which is the RHS

b Let $u=1-\sin x$, $\dfrac{du}{dx}=-\cos x$.

$$\therefore \quad \int \frac{\cos x}{1-\sin x}\,dx = -\int \frac{-\cos x}{1-\sin x}\,dx$$

$$=-\int \frac{1}{u}\frac{du}{dx}\,dx$$

$$=-\int \frac{1}{u}\,du$$

$$=-\ln|u|+c$$

$$=-\ln(1-\sin x)+c \quad \{\sin x<1\}$$

c From **a**,

$$\int \sec x+\tan x\,dx=\int \frac{\cos x}{1-\sin x}\,dx$$

$$\therefore \quad \int \sec x\,dx=\int \frac{\cos x}{1-\sin x}\,dx-\int \tan x\,dx$$

$$=-\ln(1-\sin x)-\int \frac{\sin x}{\cos x}\,dx$$

$$=-\ln(1-\sin x)+\int \frac{-\sin x}{\cos x}\,dx$$

$$=\ln|\cos x|-\ln(1-\sin x)+c$$

$$=\ln\left(\frac{\cos x}{1-\sin x}\right)+c$$

{as $\cos x>0$ for $0<x<\frac{\pi}{2}$}

$=\ln(\sec x+\tan x)+c$ {from **a**}

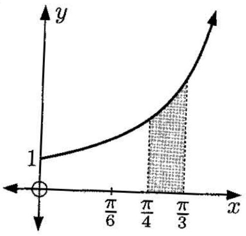
d Area

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x\,dx$$

$$=\Big[\ln(\sec x+\tan x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=\ln(2+\sqrt{3})-\ln(\sqrt{2}+1)$$

$$=\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right) \text{ units}^2$$



22 $3x^2-2y^2=10$

$$\therefore \quad 6x-4y\frac{dy}{dx}=0$$

Now when $x=2$, $3(4)-2y^2=10$

$$\therefore \quad 2y^2=2$$

$$\therefore \quad y=-1 \quad \{\text{since } y<0\}$$

At $(2,-1)$, $6(2)-4(-1)\dfrac{dy}{dx}=0$

$$\therefore \quad 4\frac{dy}{dx}=-12$$

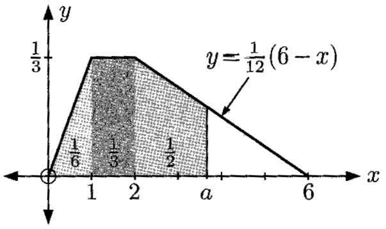
$$\therefore \quad \frac{dy}{dx}=-3$$

$\therefore$ the tangent has equation $y-(-1)=-3(x-2)$

$$\therefore \quad y+1=-3x+6$$

$$\therefore \quad y=-3x+5$$

23 **a** Since f is a density function the area between graph and the x-axis is 1.



Since area from 0 to 2 is $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$, we have to find a so that the area between the graph and the x-axis for $2\leqslant x\leqslant a$ is $\frac{1}{2}$.

$$\therefore \quad \int_2^a \tfrac{1}{12}(6-x)\,dx=\tfrac{1}{2}$$

$$\therefore \quad \int_2^a (6-x)\,dx=6$$

$$\therefore \quad \left[6x-\frac{x^2}{2}\right]_2^a=6$$

$$\therefore \quad \left(6a-\frac{a^2}{2}\right)-(12-2)=6$$

$$\therefore \quad -\frac{a^2}{2}+6a-16=0$$

$$\therefore \quad a^2-12a+32=0$$

$$\therefore \quad (a-4)(a-8)=0$$

$$\therefore \quad a=4 \text{ or } 8$$

But if $a=8$, the graph lies below the x-axis.
So, $a=4$ is the only solution.

b The median value of X is the point m for which $\text{P}(X\leqslant m)=\text{P}(X\geqslant m)=\frac{1}{2}$
From the above discussion $m=2$.

24 $6x^2+4xy+2y^2=3 \quad .... (1)$

$$\therefore \quad 12x+4y+4x\frac{dy}{dx}+4y\frac{dy}{dx}=0$$

$$\therefore \quad (4x+4y)\frac{dy}{dx}=-12x-4y$$

$$\therefore \quad \frac{dy}{dx}=\frac{-3x-y}{x+y}$$

$$\therefore \quad \frac{dy}{dx}=0 \text{ when } y=-3x$$

Substituting into (1), $6x^2+4x(-3x)+2(-3x)^2=3$

$$\therefore \quad 6x^2-12x^2+18x^2=3$$

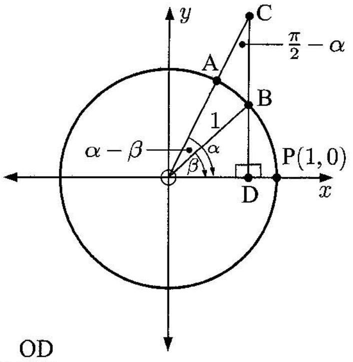
$$\therefore \quad 12x^2=3$$

$$\therefore \quad x^2=\tfrac{1}{4}$$

$$\therefore \quad x=\pm\tfrac{1}{2}$$

When $x=\frac{1}{2}$, $y=-\frac{3}{2}$ and when $x=-\frac{1}{2}$, $y=\frac{3}{2}$.
$\therefore$ the points are $(\frac{1}{2},-\frac{3}{2})$ and $(-\frac{1}{2},\frac{3}{2})$.

25



a In $\triangle$ODB:

- $\cos\beta = \dfrac{\text{OD}}{\text{OB}} = \dfrac{\text{OD}}{1}$

  $\therefore\ \text{OD} = \cos\beta$
- $\sin\beta = \dfrac{\text{BD}}{\text{OB}} = \dfrac{\text{BD}}{1}$

  $\therefore\ \text{BD} = \sin\beta$

b In $\triangle$ODC, $\cos\alpha = \dfrac{\text{OD}}{\text{OC}}$, so $\text{OC} = \dfrac{\text{OD}}{\cos\alpha}$

$\therefore\ \text{OC} = \dfrac{\cos\beta}{\cos\alpha}$ {from **a**}

c $\tan\alpha = \dfrac{\text{DC}}{\text{OD}}$

$\therefore\ \text{DC} = \text{OD}\tan\alpha = \cos\beta\tan\alpha$

$\therefore\ \text{BC} = \text{DC} - \text{BD} = \cos\beta\tan\alpha - \sin\beta$

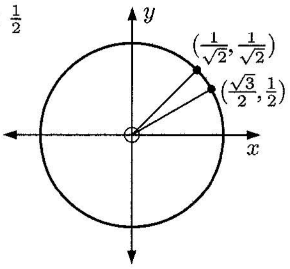
d By the sine rule in $\triangle$OBC,

$$\frac{\sin(\alpha-\beta)}{\text{BC}} = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{1}$$

$$\begin{aligned}\therefore\ \sin(\alpha-\beta) &= \text{BC}\sin\left(\tfrac{\pi}{2}-\alpha\right)\\ &= \text{BC}\cos\alpha\\ &= (\cos\beta\tan\alpha - \sin\beta)\cos\alpha \quad \{\text{from } \mathbf{c}\}\\ &= \left(\cos\beta\frac{\sin\alpha}{\cos\alpha} - \sin\beta\right)\cos\alpha\\ &= \cos\beta\sin\alpha - \sin\beta\cos\alpha\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

e

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ \quad \{\text{from } \mathbf{d}\}\\ &= \tfrac{1}{\sqrt{2}}\times\tfrac{\sqrt{3}}{2} - \tfrac{1}{\sqrt{2}}\times\tfrac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$



SOLUTIONS TO EXAMINATION PRACTICE SET 4

1 **a** $3+8+13+\ldots.$ is arithmetic with $u_1 = 3$ and $d = 5$

Thus
$$\begin{aligned}S_n &= \frac{n}{2}[2u_1 + (n-1)d]\\ &= \frac{n}{2}[6 + 5(n-1)]\\ &= \frac{n}{2}[5n+1]\\ &= \frac{n(5n+1)}{2}\end{aligned}$$

b For $S_n > 1000$, $n(5n+1) > 2000$

Using technology, $n > 19.9$

and so the smallest integer n is $n = 20$.

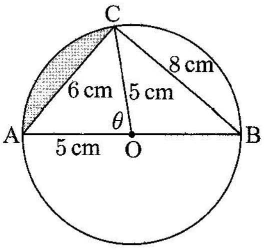
2 $(1+ax)^n$

$$\begin{aligned}&= 1^n + \tbinom{n}{1}1^{n-1}(ax) + \tbinom{n}{2}1^{n-2}(ax)^2 + \ldots.\\ &= 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \ldots.\end{aligned}$$

Thus $na = 35$ and $\dfrac{n(n-1)}{2}a^2 = 525$

$$\begin{aligned}\therefore\ (n^2-n)a^2 &= 1050\\ \therefore\ n^2a^2 - a(na) &= 1050\\ \therefore\ 1225 - 35a &= 1050\\ \therefore\ 35a &= 175\\ \therefore\ a = 5 \text{ and } n &= 7\end{aligned}$$

3

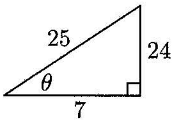


$\text{AB} = \sqrt{6^2+8^2} = \sqrt{100} = 10$ cm

$\therefore\ \text{AO} = \text{OB} = 5$ cm

By the cosine rule, $\cos\theta = \dfrac{5^2+5^2-6^2}{2\times5\times5} = \frac{14}{50} = \frac{7}{25}$

So, $\sin\theta = \frac{24}{25}$



$$\begin{aligned}\therefore\ \text{area of segment} &= \tfrac{1}{2}r^2(\theta - \sin\theta)\\ &= \tfrac{1}{2}\times 25\left(\arccos\left(\tfrac{7}{25}\right) - \tfrac{24}{25}\right)\\ &\approx 4.09\ \text{cm}^2\end{aligned}$$

4 **a** Since $\frac{1}{4} + \frac{3}{7} + k = 1$,

$k = 1 - \left(\frac{1}{4} + \frac{3}{7}\right) = \frac{9}{28}$

x	0	2	7
$P(x)$	$\frac{7}{28}$	$\frac{12}{28}$	$\frac{9}{28}$

b Mean $\mu = 0\times\frac{7}{28} + 2\times\frac{12}{28} + 7\times\frac{9}{28} = \frac{87}{28} = 3\frac{3}{28}$,

median is 2, mode is 2

c Variance $\sigma^2 = 0^2\left(\frac{7}{28}\right) + 2^2\left(\frac{12}{28}\right) + 7^2\left(\frac{9}{28}\right) - \left(\frac{87}{28}\right)^2$

≈ 7.810

and standard deviation $\sigma \approx \sqrt{7.810} \approx 2.795$

5 **a** If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $y = \frac{1}{2}$ and $\dfrac{dy}{dx} = \frac{\sqrt{3}}{2}$

$\therefore$ the normal has gradient $-\frac{2}{\sqrt{3}}$, and the equation of the normal is $y - \frac{1}{2} = -\frac{2}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$.

b The normal meets the x-axis when $y = 0$

$\therefore\ -\frac{1}{2} = -\frac{2}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$

$\therefore\ x - \frac{\pi}{6} = \frac{\sqrt{3}}{4}$

$\therefore\ x = \frac{\sqrt{3}}{4} + \frac{\pi}{6}$

$\therefore$ the coordinates are $(\frac{\sqrt{3}}{4} + \frac{\pi}{6}, 0)$.

6 As $f(x) = \dfrac{ax+2}{x^2+1}$,
$$\begin{aligned}f'(x) &= \frac{a(x^2+1) - (ax+2)2x}{(x^2+1)^2}\\ &= \frac{ax^2 + a - 2ax^2 - 4x}{(x^2+1)^2}\\ &= \frac{-ax^2 - 4x + a}{(x^2+1)^2}\end{aligned}$$

Now $f(x)$ has stationary points where $f'(x) = 0$

$$\therefore\ -ax^2 - 4x + a = 0$$

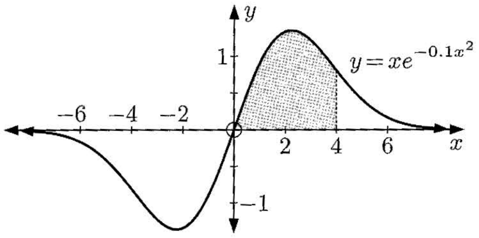
$$\therefore\ ax^2 + 4x - a = 0$$

This equation has real solutions when $\Delta = 16 - 4(a)(-a) \geqslant 0$

$$\therefore\ 16 + 4a^2 \geqslant 0$$

and this is true for all $a \in \mathbb{R}$.

7 **a**



$$\text{Area} = \int_0^4 xe^{-0.1x^2}\,dx$$

Let $u = -0.1x^2$, $\dfrac{du}{dx} = -0.2x$.

When $x = 0$, $u = 0$. When $x = 4$, $u = -1.6$.

$$\text{So, area} = \int_0^4 e^u\left(-5\,\frac{du}{dx}\right)dx$$

$$= -5\textstyle\int_0^{-1.6} e^u\,du$$

$$= -5\left[e^u\right]_0^{-1.6}$$

$$= -5e^{-1.6} + 5e^0$$

$$= 5 - 5e^{-1.6}\ \text{units}^2$$

b Volume $= \pi\displaystyle\int_0^4 y^2\,dx = \pi\int_0^4 x^2e^{-0.2x^2}\,dx$

≈ 14.1 units^3 {using technology}

8 $a = 3t^2 - 2t + 1$

Now $v = \int a\,dt = \dfrac{3t^3}{3} - \dfrac{2t^2}{2} + t + c$

$\therefore\ v = t^3 - t^2 + t + c$

But $v(0) = 5$, so $5 = c$

Thus $v = t^3 - t^2 + t + 5\ \text{m s}^{-1}$

Now $s = \int v\,dt = \dfrac{t^4}{4} - \dfrac{t^3}{3} + \dfrac{t^2}{2} + 5t + d$

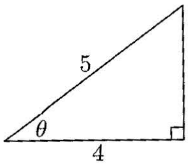
But $s(0) = 0$, so $d = 0$

$\therefore\ s = \dfrac{t^4}{4} - \dfrac{t^3}{3} + \dfrac{t^2}{2} + 5t$

9 Let $\arcsin\left(\frac{3}{5}\right) = \theta$

$\therefore\ \sin\theta = \frac{3}{5}$

and so $\cos\theta = \frac{4}{5}$



Now $\sin 2\left(\arcsin\frac{3}{5}\right) = \sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

$= \frac{24}{25}$

10 **a** **i** $\overrightarrow{BA}$

$= \overrightarrow{BO} + \overrightarrow{OA}$

$= -\mathbf{b} + \mathbf{a}$

ii $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= \mathbf{b} + 2\mathbf{a} + k\overrightarrow{BA}$

$= \mathbf{b} + 2\mathbf{a} + k(-\mathbf{b} + \mathbf{a})$

$= (1-k)\mathbf{b} + (2+k)\mathbf{a}$

b If $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{AB}$ then $\overrightarrow{OD} \bullet \overrightarrow{AB} = 0$

$\therefore\ [(2+k)\mathbf{a}\ +\ (1-k)\mathbf{b}] \bullet [\mathbf{a} - \mathbf{b}] = 0$

$\therefore\ (2+k)\mathbf{a}\bullet\mathbf{a} - (2+k)\mathbf{a}\bullet\mathbf{b}$

$+ (1-k)\mathbf{b}\bullet\mathbf{a} - (1-k)\mathbf{b}\bullet\mathbf{b} = 0$

But $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(60°) = \frac{1}{2}\,|\mathbf{a}|\,|\mathbf{b}|$

$\therefore\ (2+k)\,|\mathbf{a}|^2 - \frac{1}{2}(1+2k)\,|\mathbf{a}|\,|\mathbf{b}| - (1-k)\,|\mathbf{b}|^2 = 0$

But $|\mathbf{b}| = 3\,|\mathbf{a}|$

$\therefore\ (2+k)\,|\mathbf{a}|^2 - \frac{1}{2}(1+2k)3\,|\mathbf{a}|^2 - (1-k)9\,|\mathbf{a}|^2 = 0$

$\therefore\ \left(2 + k - \frac{3}{2} - 3k - 9 + 9k\right)|\mathbf{a}|^2 = 0$

$\therefore\ \left(-\frac{17}{2} + 7k\right)|\mathbf{a}|^2 = 0$

$\therefore\ -\frac{17}{2} + 7k = 0$ or $k = \frac{17}{14}$

11 $y = \ln(x^2 - 3)$ has $\dfrac{dy}{dx} = \dfrac{2x}{x^2 - 3}$ and

$$\frac{d^2y}{dx^2} = \frac{2(x^2-3) - 2x(2x)}{(x^2-3)^2} = \frac{2x^2 - 6 - 4x^2}{(x^2-3)^2} = \frac{-2x^2 - 6}{(x^2-3)^2}$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{-2x^2 - 6}{(x^2-3)^2} + \frac{4x^2}{(x^2-3)^2} = \frac{2x^2 - 6}{(x^2-3)^2} = \frac{2(x^2-3)}{(x^2-3)^2} = \frac{2}{x^2 - 3}$$

12 $f(x) = \dfrac{e^{x^2}}{e^x - 1}$

a The vertical asymptotes occur when $e^x - 1 = 0$

$\therefore\ e^x = 1$

$\therefore\ x = 0$

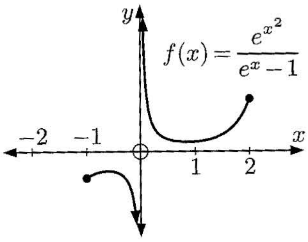
$\therefore$ the vertical asymptote is $x = 0$

b From technology:

the maximum turning point is $(-0.604, -3.18)$,

the minimum turning point is $(0.864, 1.54)$.

c



13 $\mathbf{a}$ is perpendicular to $\mathbf{b}$ if $\mathbf{a}\bullet\mathbf{b} = 0$

$\therefore\ (-2\mathbf{i} + p\mathbf{j} - \mathbf{k})\bullet(\mathbf{i} + 3(p+4)\mathbf{j} + (2p-5)\mathbf{k}) = 0$

$\therefore\ -2 + p(3p + 12) - (2p - 5) = 0$

$\therefore\ 3p^2 + 10p + 3 = 0$

$\therefore\ (3p+1)(p+3) = 0$

$\therefore\ p = -\frac{1}{3}$ or $p = -3$

14 Applying the binomial theorem to $\left(\frac{5}{2}x^2 - \dfrac{2}{5x}\right)^{12}$,

$T_{r+1} = \binom{12}{r}\left(\frac{5}{2}x^2\right)^{12-r}\left(\dfrac{-2}{5x}\right)^r$

$= \binom{12}{r}\left(\frac{5}{2}\right)^{12-r}x^{24-2r}\left(-\frac{5}{2}\right)^{-r}x^{-r}$

$= \binom{12}{r}(-1)^{-r}\left(\frac{5}{2}\right)^{12-2r}x^{24-3r}$

The term independent of x will have $24 - 3r = 0$

$\therefore\ r = 8$

So, the constant term is $\binom{12}{8}(-1)^{-8}\left(\frac{5}{2}\right)^{-4} = \binom{12}{8}\left(\frac{2}{5}\right)^4$

$= \frac{1584}{125}$

15 **a** Since f is a density function

$$\int_0^2 \left(\frac{x}{8}+c\right) dx = 1$$

$$\therefore \left[\frac{x^2}{16}+cx\right]_0^2 = 1$$

$$\therefore \tfrac{1}{4}+2c = 1$$

$$\therefore c = \tfrac{3}{8}$$

Thus, $f(x) = \frac{1}{8}(x+3)$

b $\mathrm{E}(X)$

$$= \int_0^2 x\,\tfrac{1}{8}(x+3)\,dx$$

$$= \tfrac{1}{8}\int_0^2 (x^2+3x)\,dx$$

$$= \tfrac{1}{8}\left[\frac{x^3}{3}+\frac{3x^2}{2}\right]_0^2$$

$$= \tfrac{1}{8}(\tfrac{8}{3}+6) - 0$$

$$= \tfrac{13}{12}$$

c $\mathrm{E}(X^2) = \int_0^2 x^2\,\frac{1}{8}(x+3)\,dx$

$$= \tfrac{1}{8}\int_0^2 (x^3+3x^2)\,dx$$

$$= \tfrac{1}{8}\left[\frac{x^4}{4}+\frac{3x^3}{3}\right]_0^2$$

$$= \tfrac{1}{8}(12-0) = \tfrac{3}{2}$$

Thus $\sigma^2 = \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}(X^2)-\mu^2$

$$\therefore \sigma^2 = \tfrac{3}{2} - \left(\tfrac{13}{12}\right)^2 \approx 0.326\,39$$

$$\therefore \sigma \approx 0.571$$

16
$$\frac{z}{w} = \left(\frac{5-3i}{b+2i}\right)\left(\frac{b-2i}{b-2i}\right)$$

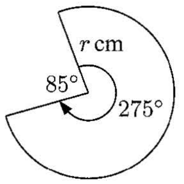
$$= \frac{5b-6+(-3b-10)i}{b^2+4}$$

$$= \left(\frac{5b-6}{b^2+4}\right) + \left(\frac{-3b-10}{b^2+4}\right)i$$

So, $\frac{z}{w}$ is real when $\left(\frac{-3b-10}{b^2+4}\right) = 0$

$$\therefore b = -\tfrac{10}{3}$$

17



The perimeter

$$= 2r + r\left(275 \times \tfrac{\pi}{180}\right)$$

$$= r\left(2+\tfrac{275\pi}{180}\right)$$

Thus $r\left(2+\frac{275\pi}{180}\right) = 100$

$$\therefore r = \frac{100}{\left(2+\frac{275\pi}{180}\right)} \approx 14.7066$$

$$\therefore \text{area} = \tfrac{1}{2}r^2\theta$$

$$\approx \tfrac{1}{2}(14.7066)^2\left(\tfrac{275\pi}{180}\right)$$

$$\approx 519 \text{ cm}^2$$

18
$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$$

$$= \int (1-\cos^2 x)\sin x\,dx$$

$$= \int \sin x - \int \cos^2 x \sin x\,dx$$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$\therefore \int \sin^3 x\,dx = -\cos x - \int u^2\left(-\frac{du}{dx}\right)dx$$

$$= -\cos x + \int u^2\,du$$

$$= -\cos x + \frac{u^3}{3} + c$$

$$= -\cos x + \tfrac{1}{3}\cos^3 x + c$$

19 **a** Let X be the number of correct answers. $X \sim \mathrm{B}(30, \frac{1}{5})$.

i $\mathrm{P}(X=20)$

$$= \binom{30}{20}\left(\tfrac{1}{5}\right)^{20}\left(\tfrac{4}{5}\right)^{10}$$

$$= 3.4 \times 10^{-8}$$

$$\approx 0$$

ii $\mathrm{P}(X \geqslant 15)$

$$= 1 - \mathrm{P}(X \leqslant 14)$$

$$\approx 1 - 0.999\,76$$

$$\approx 0.000\,24$$

iii $\mathrm{P}(X \leqslant 25) \approx 0.9999 \approx 1$

b Now $X \sim \mathrm{B}(30,\ 0.85)$.

i $\mathrm{P}(X=20)$

$$= \binom{30}{20}(0.85)^{20}(0.15)^{10}$$

$$\approx 0.006\,72$$

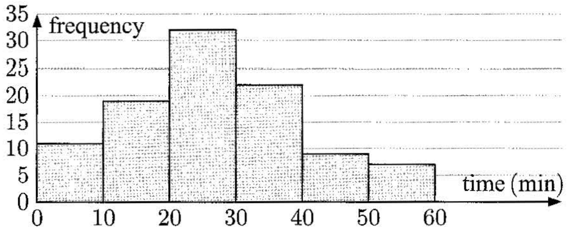
ii $\mathrm{P}(X \geqslant 15)$

$$= 1 - \mathrm{P}(X \leqslant 14)$$

$$= 1 - 1.14 \times 10^{-6}$$

$$\approx 1$$

20 **a**



b $20 \leqslant t < 30$ min

c

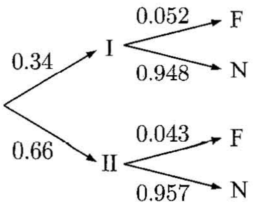
Travelling time (t min)	*Frequency* (f)	*Midpoint* (x)	fx
$0 \leqslant t < 10$	11	5	55
$10 \leqslant t < 20$	19	15	285
$20 \leqslant t < 30$	32	25	800
$30 \leqslant t < 40$	22	35	770
$40 \leqslant t < 50$	9	45	405
$50 \leqslant t < 60$	7	55	385
Total	$n = 100$		$\sum fx = 2700$

$$\mu = \frac{\sum fx}{n} = \frac{2700}{100} = 27$$

$\therefore$ mean ≈ 27 min

standard deviation ≈ 13.4 min {technology}

21 The information is displayed in the tree diagram.



a P(an item is faulty)

$= 0.34 \times 0.052 + 0.66 \times 0.043 = 0.046\,06$

b Let II be the event of machine II, F be the event of a faulty item.

$$\therefore \mathrm{P}(\mathrm{II} \mid F) = \frac{\mathrm{P}(\mathrm{II} \cap F)}{\mathrm{P}(F)}$$

$$= \frac{0.66 \times 0.043}{0.046\,06}$$

$$\approx 0.616$$

22 **a** Let X be the weight, then $X \sim \mathrm{N}(120,\ 1.063^2)$.

Hence, $\mathrm{P}(X < 118) \approx 0.029\,95$

So, 2.995% of the bags are rejected.

b We now have $X \sim \mathrm{N}(120,\ \sigma^2)$.

Since $\mathrm{P}(X < 118) = 0.06$,

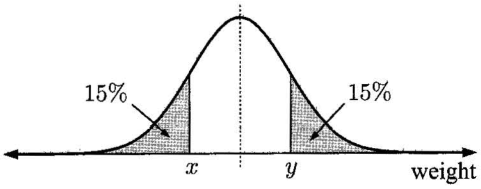
$$\mathrm{P}\left(\frac{X-120}{\sigma} < \frac{118-120}{\sigma}\right) = 0.06$$

$$\therefore\ \mathrm{P}\left(Z < \frac{-2}{\sigma}\right) = 0.06$$
$$\therefore\ \frac{-2}{\sigma} \approx -1.555$$
$$\therefore\ \sigma \approx 1.286$$

c X is now $\sim \mathrm{N}(\mu,\ 1.286^2)$.
Since $\mathrm{P}(X < 118) = 0.03$,

$$\mathrm{P}\left(\frac{X-\mu}{1.286} < \frac{118-\mu}{1.286}\right) = 0.03$$
$$\therefore\ \mathrm{P}\left(Z < \frac{118-\mu}{1.286}\right) = 0.03$$
$$\therefore\ \frac{118-\mu}{1.286} \approx -1.8808$$
$$\therefore\ \mu \approx 120.42$$

d



Since $\mathrm{P}(x \leqslant X \leqslant y) = 0.70$, by symmetry,
$\mathrm{P}(X \leqslant x) = \mathrm{P}(X \geqslant y) = 0.15$.
But $X \sim \mathrm{N}(120.42,\ 1.286^2)$.
For $\mathrm{P}(X \leqslant x) = 0.15$, $x \approx 119.09$

Using the symmetry, $\dfrac{x+y}{2} = 120.42$

$$\therefore\ y \approx 240.84 - 119.09$$
$$\approx 121.75$$

23 **a** $f(0) = \frac{-2}{-6} = \frac{1}{3}$

$\therefore$ the y-intercept is $\frac{1}{3}$.
When $y = 0$, $x - 2 = 0$
$\therefore\ x = 2$
$\therefore$ the x-intercept is 2.

b
$$f'(x) = \frac{1(x^2+cx-6) - (x-2)(2x+c)}{(x^2+cx-6)^2}$$
$$= \frac{x^2+cx-6-2x^2+4x-cx+2c}{(x^2+cx-6)^2}$$
$$= \frac{-x^2+4x+2c-6}{(x^2+cx-6)^2}$$

c It has at least one stationary point when
$-x^2+4x+2c-6 = 0$ has real roots

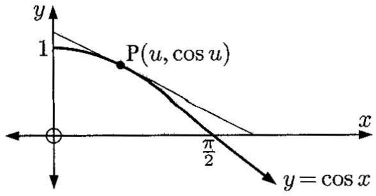
$$\therefore\ \Delta \geqslant 0$$
$$\therefore\ 16 - 4(-1)(2c-6) \geqslant 0$$
$$\therefore\ 16 + 8c - 24 \geqslant 0$$
$$\therefore\ 8c \geqslant 8 \text{ and so } c \geqslant 1.$$

d When $c = 0$, $f'(x) = \dfrac{-x^2+4x-6}{(x^2-6)^2}$

Using technology, $f''(x) = 0$ when $x \approx 0.893$
$\therefore$ there is a point of inflection at $(0.893, 0.213)$.

e
$$\int_{-1}^{1} \frac{x-2}{x^2-4x-6}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{2x-4}{x^2-4x-6}\,dx$$
$$= \tfrac{1}{2}\left[\ln|x^2-4x-6|\right]_{-1}^{1}$$
$$= \tfrac{1}{2}\left[\ln 9 - \ln 1\right]$$
$$= \tfrac{1}{2}\ln 3^2$$
$$= \ln 3$$

24 **a**



$$\frac{dy}{dx} = -\sin x = -\sin u \quad \text{at} \quad (u, \cos u)$$

$\therefore$ tangent has equation

$$y - \cos u = -\sin u(x - u)$$
$$\therefore\ y = (-\sin u)x + u\sin u + \cos u$$

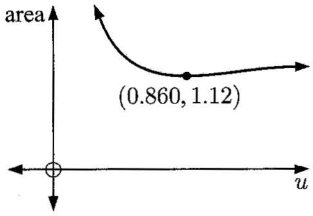
b The tangent cuts the x-axis when $y = 0$

$$\therefore\ (\sin u)x = u\sin u + \cos u$$
$$\therefore\ x = u + \cot u$$

$\therefore$ A is at $(u + \cot u,\ 0)$
The tangent cuts the y-axis when $x = 0$
$\therefore\ y = u\sin u + \cos u$
$\therefore$ B is at $(0,\ u\sin u + \cos u)$.

c
$$\text{Area} = \tfrac{1}{2}(\mathrm{AO})(\mathrm{OB})$$
$$= \tfrac{1}{2}\left(u + \frac{\cos u}{\sin u}\right)(u\sin u + \cos u)$$
$$= \tfrac{1}{2}\left(\frac{u\sin u + \cos u}{\sin u}\right)(u\sin u + \cos u)$$
$$= \frac{(u\sin u + \cos u)^2}{2\sin u} \text{ units}^2$$

d Using technology:



So, the area is minimised when $u \approx 0.860$.

P is $\approx (0.860,\ 0.652)$.

25 **a** Let θ be the acute angle between the lines, then

$$\left|\begin{pmatrix}1\\2\\-1\end{pmatrix} \bullet \begin{pmatrix}2\\-1\\1\end{pmatrix}\right| = \sqrt{1+4+1}\sqrt{4+1+1}\cos\theta$$
$$\therefore\ |2-2-1| = \sqrt{6}\sqrt{6}\cos\theta$$
$$\therefore\ \cos\theta = \tfrac{1}{6} \text{ and so } \theta \approx 80.4°$$

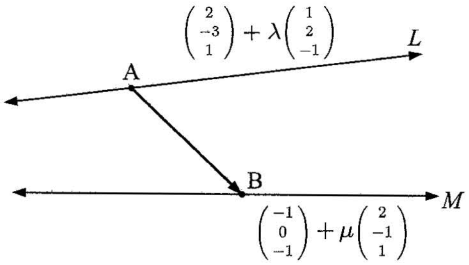
b A vector perpendicular to both lines is

$$\begin{pmatrix}1\\2\\-1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix}$$
$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -1\\ 2 & -1 & 1\end{vmatrix}$$
$$= (2-1)\mathbf{i} - (1 - -2)\mathbf{j} + (-1-4)\mathbf{k}$$
$$= \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

c Since $(2, -3, 1)$ is a point on the line, an equation is

$$x - 3y - 5z = 2 - 3(-3) - 5(1)$$
$$\therefore\ x - 3y - 5z = 6$$

d



$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -1 - \lambda + 2\mu \\ 3 - 2\lambda - \mu \\ -2 + \lambda + \mu \end{pmatrix}$$

e $\overrightarrow{AB}$ is parallel to $\mathbf{n}$ if $\begin{pmatrix} -1 - \lambda + 2\mu \\ 3 - 2\lambda - \mu \\ -2 + \lambda + \mu \end{pmatrix} = k \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$

This gives the three equations
$$\begin{aligned} -\lambda + 2\mu - k &= 1 \\ -2\lambda - \mu + 3k &= -3 \\ \lambda + \mu + 5k &= 2 \end{aligned}$$

This has unique solution

$\lambda = 1, \quad \mu = 1, \quad k = 0,$ and $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The distance between the lines L and M is zero.

Also, as $\lambda = 1$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

So, the lines L and M intersect at the point $(3, -1, 0)$.

SOLUTIONS TO EXAMINATION PRACTICE SET 5

1 **a**
$$\begin{aligned} (a + 2i)(b - i) &= 17 + 7i \\ \therefore\ ab + 2 = 17 &\text{ and } 2b - a = 7 \\ \therefore\ ab = 15 &\text{ and } 2b - a = 7 \\ \therefore\ (2b - 7)b &= 15 \\ \therefore\ 2b^2 - 7b - 15 &= 0 \\ \therefore\ (2b + 3)(b - 5) &= 0 \\ \therefore\ b &= -\tfrac{3}{2} \text{ or } 5 \end{aligned}$$

When $b = -\frac{3}{2}$, $a = -10$, and when $b = 5$, $a = 3$.

So, $a = 3$, $b = 5$ or $a = -10$, $b = -\frac{3}{2}$.

b
$$\begin{aligned} (p + qi)^2 &= -3 + 6i\sqrt{6} \\ \therefore\ p^2 - q^2 + 2pqi &= -3 + 6\sqrt{6}i \\ \therefore\ p^2 - q^2 = -3 &\text{ and } pq = 3\sqrt{6} \\ \therefore\ p^2 - \frac{54}{p^2} + 3 = 0 & \quad \left\{\text{using } q = \frac{3\sqrt{6}}{p}\right\} \\ \therefore\ p^4 + 3p^2 - 54 &= 0 \end{aligned}$$

$\therefore\ (p^2 + 9)(p^2 - 6) = 0$ and so $p^2 = -9$ or 6

But p is real and > 0 $\therefore\ p = \sqrt{6}$

So, $p = \sqrt{6}$, $q = 3$.

2
$$\begin{aligned} 2\sin\left(x + \tfrac{\pi}{6}\right) &= \sin x \\ \therefore\ 2\left[\sin x \cos \tfrac{\pi}{6} + \cos x \sin \tfrac{\pi}{6}\right] &= \sin x \\ \therefore\ 2\sin x \times \tfrac{\sqrt{3}}{2} + 2\cos x \times \tfrac{1}{2} &= \sin x \\ \therefore\ \sqrt{3}\sin x + \cos x &= \sin x \\ \therefore\ \sin x(\sqrt{3} - 1) &= -\cos x \\ \therefore\ \frac{\sin x}{\cos x} &= \frac{-1}{\sqrt{3} - 1} \\ \therefore\ \tan x &= \left(\frac{-1}{\sqrt{3} - 1}\right)\left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\ &= \frac{-\sqrt{3} - 1}{3 - 1} \\ &= -\tfrac{1}{2}(1 + \sqrt{3}) \end{aligned}$$

3
$$\begin{aligned} \frac{6}{7^x} - 2 \times 7^x &= 1 \\ \therefore\ 6 - 2 \times (7^x)^2 &= 7^x \\ \therefore\ 2(7^x)^2 + 7^x - 6 &= 0 \end{aligned}$$

Let $7^x = m$, so $2m^2 + m - 6 = 0$
$$\begin{aligned} \therefore\ (2m - 3)(m + 2) &= 0 \\ \therefore\ m &= \tfrac{3}{2} \text{ or } -2 \end{aligned}$$

Thus $7^x = \frac{3}{2}$ $\{7^x > 0$ for all $x\}$

Hence $x = \dfrac{\log \frac{3}{2}}{\log 7}$ or $\log_7 \frac{3}{2}$.

4 **a**
$$\begin{aligned} y &= (x - 3)e^{2x} \\ \therefore\ \frac{dy}{dx} &= 1e^{2x} + (x - 3)e^{2x}(2) \\ &= e^{2x}(1 + 2x - 6) \\ &= e^{2x}(2x - 5) \end{aligned}$$

So, $\dfrac{dy}{dx} = 0$ when $2x - 5 = 0$ $\therefore\ x = \frac{5}{2}$

$\therefore$ when $x = \frac{5}{2}$ we have a stationary point.

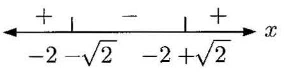
b
$$\begin{aligned} f(x) &= x^2e^x \\ \text{has } f'(x) &= 2xe^x + x^2e^x = e^x(2x + x^2) \\ \text{and } f''(x) &= e^x(2x + x^2) + e^x(2 + 2x) \\ &= e^x(x^2 + 4x + 2) \end{aligned}$$

The graph of $y = f(x)$ is concave down when $f''(x) < 0$
$\therefore\ x^2 + 4x + 2 < 0$

$x^2 + 4x + 2$ has critical values

$$\frac{-4 \pm \sqrt{16 - 4(1)(2)}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

and has sign diagram:



$\therefore$ $f(x)$ is concave down for $x \in \left]-2 - \sqrt{2}, -2 + \sqrt{2}\right[$.

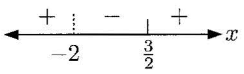
5 **a** $x^2 - 1 \geqslant 0$ $\therefore$ $x^2 \geqslant 1$ $\therefore$ $x \leqslant -1$ or $x \geqslant 1$

b $1 - x^2 > 0$ $\therefore$ $x^2 < 1$ $\therefore$ $-1 < x < 1$

c $x \neq 0$

d $x \neq -2$ and $\dfrac{2x - 3}{x + 2} \geqslant 0$

$\dfrac{2x - 3}{x + 2}$ has sign diagram:



$G(x)$ takes on real values if $x < -2$ or $x \geqslant \frac{3}{2}$.

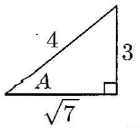
6 $g(x) = 2x^2 - 3x + \sin x$

f is a reflection of g in the x-axis.

$\therefore$ $f(x) = -g(x) = -2x^2 + 3x - \sin x$

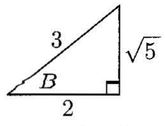
h is a reflection of g in the y-axis.
$$\begin{aligned} \therefore\ h(x) &= g(-x) \\ &= 2(-x)^2 - 3(-x) + \sin(-x) \\ &= 2x^2 + 3x - \sin x \end{aligned}$$

7 $\sin A = \frac{3}{4}$



$\cos A = \frac{\sqrt{7}}{4}$

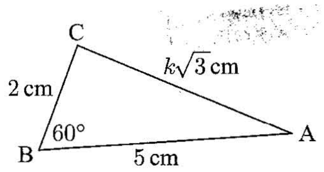
$\cos B = \frac{2}{3}$



$\sin B = \frac{\sqrt{5}}{3}$

$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B\\ &= \tfrac{3}{4}\times\tfrac{2}{3} - \tfrac{\sqrt{7}}{4}\times\tfrac{\sqrt{5}}{3}\\ &= \frac{6-\sqrt{35}}{12}\end{aligned}$$

8



$$\begin{aligned}(k\sqrt{3})^2 &= 2^2+5^2-2\times 2\times 5\times\cos 60^\circ \quad \{\text{cosine rule}\}\\ \therefore\ 3k^2 &= 4+25-10\\ \therefore\ 3k^2 &= 19\\ \therefore\ k^2 &= \tfrac{19}{3}\\ \therefore\ k &= \sqrt{\tfrac{19}{3}} \quad \{\text{as } k>0\}\end{aligned}$$

9

$$x^3y^2 - xy + y = 4$$

When $x=1$, $y^2 - y + y = 4$

$$\begin{aligned}\therefore\ y^2 &= 4\\ \therefore\ y &= 2 \quad \{\text{as } y>0\}\end{aligned}$$

So, the point of contact is $(1, 2)$.

Now $3x^2y^2 + x^3\left(2y\,\dfrac{dy}{dx}\right) - \left(1y + x\,\dfrac{dy}{dx}\right) + \dfrac{dy}{dx} = 0$

$$\begin{aligned}\therefore \text{ at } (1, 2),\quad 12 + 4\,\frac{dy}{dx} - \left(2 + \frac{dy}{dx}\right) + \frac{dy}{dx} &= 0\\ \therefore\ 10 + 4\,\frac{dy}{dx} &= 0\\ \therefore\ \frac{dy}{dx} &= -\tfrac{5}{2}\end{aligned}$$

$$\begin{aligned}\therefore \text{ the tangent has equation}\quad y-2 &= -\tfrac{5}{2}(x-1)\\ \therefore\ 2y-4 &= -5x+5\\ \therefore\ 5x+2y &= 9\end{aligned}$$

10 Let $u = x-3$, $\dfrac{du}{dx} = 1$.

$$\begin{aligned}\therefore\ \int x\sqrt{x-3}\,dx &= \int (u+3)\sqrt{u}\,\frac{du}{dx}\,dx\\ &= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du\\ &= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= \tfrac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c\end{aligned}$$

11 We integrate by parts with $u = \arcsin x \qquad v' = 1$

$$u' = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$\begin{aligned}\therefore\ \int \arcsin x\,dx &= \int 1\arcsin x\,dx\\ &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx\end{aligned}$$

Let $w = 1-x^2$, $\dfrac{dw}{dx} = -2x$

$$\begin{aligned}\therefore\ \int \arcsin x\,dx &= x\arcsin x - \int w^{-\frac{1}{2}}\left(\tfrac{1}{-2}\,\frac{dw}{dx}\right) dx\\ &= x\arcsin x + \tfrac{1}{2}\int w^{-\frac{1}{2}}\,dw\\ &= x\arcsin x + \tfrac{1}{2}\,\frac{w^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= x\arcsin x + \sqrt{1-x^2} + c\end{aligned}$$

12 When $t=0$, $s = 3e^{2(0)}$

$$= 3$$

So, the initial displacement is $s = 3$ metres.

$$\begin{aligned}\frac{ds}{dt} &= 3(2e^{2t})\\ &= 2(3e^{2t})\\ &= 2s\end{aligned}$$

So, $v = 2s$

13

$$\begin{aligned}\tan x - \frac{\tan x - 1}{\tan x + 1} &= \frac{\tan x(\tan x+1) - (\tan x - 1)}{\tan x + 1}\\ &= \frac{\tan^2 x + \tan x - \tan x + 1}{\tan x + 1}\\ &= \frac{\sec^2 x}{\tan x + 1}\end{aligned}$$

$$\therefore\ \int \tan x - \frac{\tan x - 1}{\tan x + 1}\,dx = \int \frac{\sec^2 x}{\tan x + 1}\,dx$$

Let $u = \tan x + 1$, $\dfrac{du}{dx} = \sec^2 x$

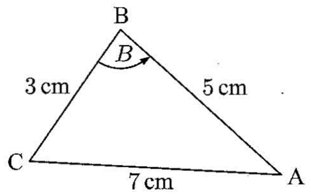
$$\begin{aligned}\therefore\ \int \tan x - \frac{\tan x - 1}{\tan x + 1}\,dx &= \int \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + c\end{aligned}$$

$$\begin{aligned}\therefore\ \int_0^{\frac{\pi}{4}} \left(\tan x - \frac{\tan x - 1}{\tan x + 1}\right) dx &= \Big[\ln|\tan x + 1|\Big]_0^{\frac{\pi}{4}}\\ &= \ln 2 - \ln 1\\ &= \ln 2\end{aligned}$$

14

$$\begin{aligned}\sin x &= \cos\left(x + \tfrac{\pi}{3}\right)\\ \therefore\ \sin x &= \cos x\cos\tfrac{\pi}{3} - \sin x\sin\tfrac{\pi}{3}\\ \therefore\ \sin x &= \cos x\left(\tfrac{1}{2}\right) - \sin x \times \tfrac{\sqrt{3}}{2}\\ \therefore\ \sin x\left(1 + \tfrac{\sqrt{3}}{2}\right) &= \tfrac{1}{2}\cos x\\ \therefore\ \frac{\sin x}{\cos x} &= \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}\\ &= \frac{1}{2+\sqrt{3}}\\ \therefore\ \tan x &= \left(\frac{1}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)\\ &= \frac{2-\sqrt{3}}{1}\\ &= 2-\sqrt{3}\end{aligned}$$

15



a
$$\begin{aligned}\cos B &= \frac{3^2+5^2-7^2}{2\times 3\times 5}\\ &= \tfrac{-15}{30}\\ &= -\tfrac{1}{2}\end{aligned}$$

b $\sin B = \frac{\sqrt{3}}{2}$

$$\begin{aligned}\text{and area} &= \tfrac{1}{2} \times 3 \times 5 \times \sin B\\ &= \tfrac{15}{2} \times \tfrac{\sqrt{3}}{2}\\ &= \tfrac{15}{4}\sqrt{3}\ \text{cm}^2\end{aligned}$$

16 $\sin\gamma = \frac{\sqrt{3}}{7}$

$$\begin{aligned}\therefore\ \cos 2\gamma &= 1 - 2\sin^2\gamma\\ &= 1 - 2\left(\tfrac{3}{49}\right)\\ &= 1 - \tfrac{6}{49} = \tfrac{43}{49}\end{aligned}$$

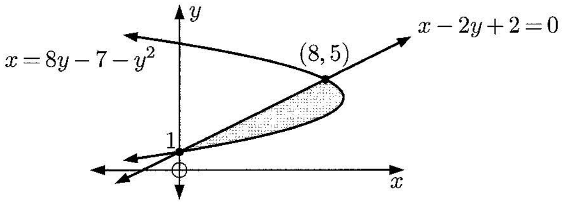
17 $x = 8y - 7 - y^2$ meets $x - 2y + 2 = 0$ where

$$\begin{aligned}8y - 7 - y^2 &= 2y - 2\\ \therefore\ y^2 - 6y + 5 &= 0\\ \therefore\ (y-1)(y-5) &= 0\\ \therefore\ y &= 1 \text{ or } 5\end{aligned}$$

When $y = 1$, $x = 2(1) - 2 = 0$

When $y = 5$, $x = 2(5) - 2 = 8$

$\therefore$ they meet at $(0, 1)$ and at $(8, 5)$



$$\begin{aligned}\text{Area} &= \int_1^5 ((8y - 7 - y^2) - (2y - 2))\, dy\\ &= \int_1^5 (-y^2 + 6y - 5)\, dy\\ &= \left[\frac{-y^3}{3} + \frac{6y^2}{2} - 5y\right]_1^5\\ &= \left(-\tfrac{125}{3} + 75 - 25\right) - \left(-\tfrac{1}{3} + 3 - 5\right)\\ &= \tfrac{32}{3}\ \text{units}^2\end{aligned}$$

18 $s = 4e^{0.2t} - e^{0.3t} + 10$ m

a $s(0) = 4 - 1 + 10 = 13$ m right of O.

b $v(t) = \dfrac{ds}{dt} = 0.8e^{0.2t} - 0.3e^{0.3t}\ \text{m s}^{-1}$

$\therefore\ v(0) = 0.8 - 0.3 = 0.5\ \text{m s}^{-1}$

c It is at rest when $v(t) = 0$

$$\begin{aligned}\therefore\ 0.8e^{0.2t} - 0.3e^{0.3t} &= 0\\ \therefore\ 0.8e^{0.2t} &= 0.3e^{0.3t}\\ \therefore\ \frac{e^{0.3t}}{e^{0.2t}} &= \frac{0.8}{0.3}\\ \therefore\ e^{0.1t} &= \tfrac{8}{3}\\ \therefore\ 0.1t &= \ln\left(\tfrac{8}{3}\right)\\ \therefore\ t &= 10\ln\left(\tfrac{8}{3}\right)\ \text{s}\end{aligned}$$

19 **a**

$$\begin{aligned}&(\cos\theta + i\sin\theta)^5\\ &= (c + is)^5 \quad \text{where } c = \cos\theta \text{ and } s = \sin\theta\\ &= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5\\ &= \left[c^5 - 10c^3s^2 + 5cs^4\right] + i\left[5c^4s - 10c^2s^3 + s^5\right]\end{aligned}$$

$$\begin{aligned}\text{But } (\cos\theta + i\sin\theta)^5 &= (\text{cis}\,\theta)^5\\ &= \text{cis}\,5\theta \quad \{\text{De Moivre}\}\\ &= \cos 5\theta + i\sin 5\theta\end{aligned}$$

Equating imaginary parts,

$$\begin{aligned}\sin 5\theta &= 5c^4s - 10c^2s^3 + s^5\\ &= 5s(1 - s^2)^2 - 10s^3(1 - s^2) + s^5\\ &= 5s(1 - 2s^2 + s^4) - 10s^3 + 10s^5 + s^5\\ &= 16s^5 - 20s^3 + 5s\\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

b If $\theta = 36°$, $5\theta = 180°$

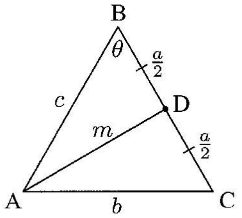
$\therefore$ $\sin 36°$ is a solution of

$$16s^5 - 20s^3 + 5s = \sin 180° = 0$$

$$\begin{aligned}\therefore\ s(16s^4 - 20s^2 + 5) &= 0\\ \therefore\ 16s^4 - 20s^2 + 5 &= 0, \quad \text{as } s \neq 0\\ \therefore\ s^2 &= \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}\\ \therefore\ s^2 &= \frac{20 \pm \sqrt{80}}{32}\\ \therefore\ s^2 &= \frac{10 \pm 2\sqrt{5}}{16}\\ \therefore\ s &= \pm\frac{\sqrt{10 \pm 2\sqrt{5}}}{4}\end{aligned}$$

So, $\sin 36° = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$.

20 **a**



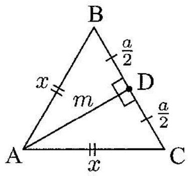
Let $\widehat{ABC} = \theta$.

Using the cosine rule in $\triangle$ABD and $\triangle$ABC:

$$\cos\theta = \frac{c^2 + \left(\frac{a}{2}\right)^2 - m^2}{2c\left(\frac{a}{2}\right)} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\begin{aligned}\therefore\ \frac{c^2 + \frac{a^2}{4} - m^2}{ac} &= \frac{a^2 + c^2 - b^2}{2ac}\\ \therefore\ 2\left(c^2 + \frac{a^2}{4} - m^2\right) &= a^2 + c^2 - b^2\\ \therefore\ 2c^2 + \frac{a^2}{2} - 2m^2 &= a^2 + c^2 - b^2\\ \therefore\ 4c^2 + a^2 - 4m^2 &= 2a^2 + 2c^2 - 2b^2\\ \therefore\ 4m^2 &= 2c^2 + 2b^2 - a^2\\ \therefore\ m^2 &= \tfrac{1}{4}(2c^2 + 2b^2 - a^2)\end{aligned}$$

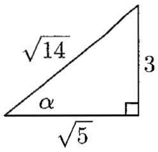
b If $x = b = c$,



$$\begin{aligned}m^2 &= \tfrac{1}{4}(4x^2 - a^2)\\ \therefore\ 4m^2 &= 4x^2 - a^2\\ \therefore\ m^2 &= \frac{4x^2 - a^2}{4}\\ \therefore\ m &= \tfrac{1}{2}\sqrt{4x^2 - a^2} \quad \{\text{since } m > 0\}\end{aligned}$$

$$\begin{aligned}\text{and area} &= \tfrac{1}{2}(a)(m)\\ &= \tfrac{1}{2}a \times \tfrac{1}{2}\sqrt{4x^2 - a^2}\\ &= \frac{a}{4}\sqrt{4x^2 - a^2}\ \text{units}^2\end{aligned}$$

21 $\tan\alpha = \frac{3}{\sqrt{5}}, \quad \pi < \alpha < \frac{3\pi}{2}$



$$\therefore \quad \sin\alpha = -\tfrac{3}{\sqrt{14}}$$

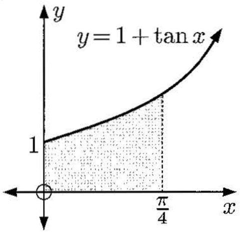
$$\cos\alpha = -\tfrac{\sqrt{5}}{\sqrt{14}}$$

So, $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(-\tfrac{3}{\sqrt{14}}\right)\left(-\tfrac{\sqrt{5}}{\sqrt{14}}\right)$$

$$= \tfrac{3\sqrt{5}}{7}$$

22



Volume

$$= \pi\int_0^{\frac{\pi}{4}} y^2\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \left(1 + 2\tan x + \tan^2 x\right) dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 2\frac{-\sin x}{\cos x}\right) dx$$

$$= \pi\left[\tan x - 2\ln|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$= \pi\left[(1 - 2\ln\tfrac{1}{\sqrt{2}}) - (0 - 0)\right]$$

$$= \pi(1 - 2(\ln 2^{-\frac{1}{2}}))$$

$$= \pi(1 + \ln 2) \text{ units}^3$$

23 **a** The number of ways the seats can be chosen is $4! = 24$.

b The order must be AABB or BBAA.

$\therefore$ the total number of possibilities is $2 \times 2 \times 2 = 8$.

c The number of ways Adam and Ben can sit next to one another is $2 \times 3! = 12$

$\therefore$ P(Adam sits next to Ben) $= \frac{12}{24} = \frac{1}{2}$

24 **a** If $\mathbf{r} = (-2t + 2)\mathbf{i} + t\mathbf{j} + (3t + 1)\mathbf{k}$ then

$$\mathbf{r} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2t+2 \\ t \\ 3t+1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= (-4t + 4) + t + (3t + 1)$$

$= 5$ and hence L_1 lies in the plane.

b If $\mathbf{r} \bullet \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix} = 3$, then $\begin{pmatrix} -2t+2 \\ t \\ 3t+1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix} = 3$

$$\therefore \quad (-2t + 2) + kt + (3t + 1) = 3$$

$$\therefore \quad t(1 + k) = 0$$

$$\therefore \quad k = -1$$

c The line L_1 lies in the plane $-2x + py + 2z = q$

if $-2(-2t + 2) + p(t) + 2(3t + 1) = q$

$$\therefore \quad 4t - 4 + pt + 6t + 2 = q$$

$$\therefore \quad (10 + p)t = q + 2$$

$\therefore$ the line L_1 lies in the plane $-2x + py + 2z = q$ if $p = -10$ and $q = -2$.

The system of equations

$$2x + y + z = 5$$
$$x - y + z = 3$$
$$-2x - 10y + 2z = -2$$

has an infinite number of solutions since the line L_1: $(-2t + 2)\mathbf{i} + t\mathbf{j} + (3t + 1)\mathbf{k}$ lies in each of the 3 planes.

25 **a** P_n is "$\cos\theta + \cos 3\theta + ... + \cos(2n - 1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$, $\sin\theta \neq 0$" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= \cos\theta$ and

$$\text{RHS} = \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$$

$\therefore$ LHS = RHS

$\therefore$ P_1 is true.

(2) If P_k is true,

$$\cos\theta + \cos 3\theta + ... + \cos(2k - 1)\theta = \frac{\sin(2k\theta)}{2\sin\theta}$$

$$\therefore \cos\theta + \cos 3\theta + ... + \cos(2k - 1)\theta + \cos(2k + 1)\theta$$

$$= \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k + 1)\theta$$

$$= \frac{\sin(2k\theta) + 2\sin\theta\cos(2k + 1)\theta}{2\sin\theta}$$

But $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

$$\therefore \quad 2\sin\theta\cos(2k + 1)\theta$$

$$= \sin(\theta + (2k + 1)\theta) + \sin(\theta - (2k + 1)\theta)$$

$$= \sin((2k + 2)\theta) + \sin(-2k\theta)$$

$$= \sin 2(k + 1)\theta - \sin(2k\theta)$$

$$\therefore \cos\theta + \cos 3\theta + ... + \cos(2k - 1)\theta + \cos(2k + 1)\theta$$

$$= \frac{\sin(2k\theta) + \sin 2(k + 1)\theta - \sin(2k\theta)}{2\sin\theta}$$

$$= \frac{\sin 2(k + 1)\theta}{2\sin\theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$ {Principle of mathematical induction}

b $(\cos\theta + i\sin\theta)^3$

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[3\cos^2\theta\sin\theta - \sin^3\theta\right]$$

c $\cos 3\theta + i\sin 3\theta$

$= \text{cis}\,3\theta$

$= (\text{cis}\,\theta)^3$ {De Moivre}

$= (\cos\theta + i\sin\theta)^3$

$= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[3\cos^2\theta\sin\theta - \sin^3\theta\right]$

Equating real parts,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

d From **a** with $n = 2$,

$$\cos\theta + \cos 3\theta = \frac{\sin 4\theta}{2\sin\theta}$$

$$\therefore \quad \sin 4\theta = 2\sin\theta(\cos\theta + \cos 3\theta)$$

$$= 2\sin\theta(4\cos^3\theta - 2\cos\theta)$$

$$= 8\sin\theta\cos^3\theta - 4\sin\theta\cos\theta$$

e $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin 6\theta}{2\sin\theta}\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos\theta + \cos 3\theta + \cos 5\theta)\,d\theta$$

$$= \left[\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(1 - \tfrac{1}{3} + \tfrac{1}{5}\right) - \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{10}\right)$$

$$= -\tfrac{1}{15}$$

SOLUTIONS TO EXAMINATION PRACTICE SET 6

1 $\ln\sqrt{2}+\ln 2+\ln\sqrt{8}+\dots$

$=\ln 2^{\frac{1}{2}}+\ln 2+\ln 2^{\frac{3}{2}}+\dots$

$=\frac{1}{2}\ln 2+\ln 2+\frac{3}{2}\ln 2+\dots$

$=\ln 2\left[\frac{1}{2}+1+\frac{3}{2}+\dots\right]$

The arithmetic series has $u_1=\frac{1}{2}$, $d=\frac{1}{2}$.

$S_n=\ln 2\left(\frac{n}{2}[2u_1+(n-1)d]\right)$

$=\ln 2\left(\frac{n}{2}[1+(n-1)\frac{1}{2}]\right)$

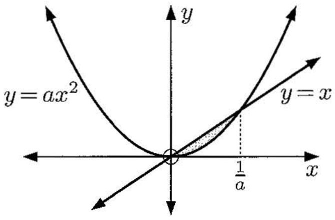
$\therefore\ S_{100}=\ln 2\left(50[1+\frac{99}{2}]\right)$

$=\ln 2\left(50\times\frac{101}{2}\right)$

$=2525\ln 2$

2 $y=ax^2$ meets $y=x$ where $ax^2=x$

$\therefore\ x(ax-1)=0$ and so $x=0$ or $\dfrac{1}{a}$



$$\text{Area}=\int_0^{\frac{1}{a}}(x-ax^2)\,dx$$

$$=\left[\frac{x^2}{2}-\frac{ax^3}{3}\right]_0^{\frac{1}{a}}$$

$$=\frac{1}{2a^2}-\frac{1}{3a^2}-0$$

$$=\frac{1}{6a^2}\text{ units}^2$$

3 We integrate by parts with $u=x^2$, $v'=e^x$; $u'=2x$, $v=e^x$

$\therefore\ \int x^2e^x\,dx=e^xx^2-\int 2xe^x\,dx$

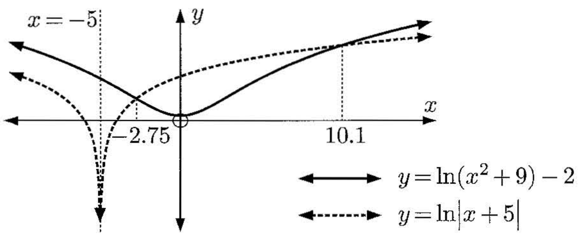
We again integrate by parts, now with $u=2x$, $v'=e^x$; $u'=2$, $v=e^x$

$\therefore\ \int x^2e^x\,dx=e^xx^2-\left[2xe^x-\int 2e^x\,dx\right]$

$=e^xx^2-2xe^x+2e^x+c$

4 $\ln(x^2+9)-2=\ln|x+5|$

We graph $y=\ln(x^2+9)-2$ and
$y=\ln|x+5|$ on the same set of axes.



$x\approx-2.75$ or 10.1

5 $Y\sim\text{B}(n,\,p)$ has mean $=np=3$, and standard deviation $=\sqrt{np(1-p)}=\frac{3}{2}$.

Thus, $np(1-p)=\frac{9}{4}$ $\therefore\ 3(1-p)=\frac{9}{4}$ $\therefore\ 1-p=\frac{3}{4}$

Hence, $p=\frac{1}{4}$ and $n=12$, and so $Y\sim\text{B}(12,\,\frac{1}{4})$.

So, $\text{P}(Y\leqslant 4)\approx 0.842$.

6 **a** $v=\dfrac{2t}{4+t^2}\text{ m s}^{-1}$

$$s=\int v\,dt=\int\frac{2t}{4+t^2}\,dt$$

$\therefore\ s=\ln\left|4+t^2\right|+c$

Since $4+t^2>0$ for all t, $s=\ln(4+t^2)+c$.

But $s(0)=-3$, so $\ln 4+c=-3$.

$\therefore\ c=-3-\ln 4$

So, $s=\ln(4+t^2)-3-\ln 4$

$\therefore\ s=\ln\left(\dfrac{t^2+4}{4}\right)-3$

b $a=\dfrac{dv}{dt}=\dfrac{2(4+t^2)-2t(2t)}{(4+t^2)^2}=\dfrac{8-2t^2}{(4+t^2)^2}\text{ m s}^{-2}$

7 The curves meet when $e^{-\frac{1}{2}x^2}=e^{\frac{1}{2}x^2}-1$

$\therefore\ e^{\frac{1}{2}x^2}-1-e^{-\frac{1}{2}x^2}=0$

$\therefore\ e^{x^2}-e^{\frac{1}{2}x^2}-1=0$ {multiply by $e^{\frac{1}{2}x^2}$}

$\therefore\ y^2-y-1=0$ {letting $y=e^{\frac{1}{2}x^2}$}

$\therefore\ y=\dfrac{1\pm\sqrt{1+4}}{2}$

$\therefore\ e^{\frac{1}{2}x^2}=\dfrac{1\pm\sqrt{5}}{2}$

$\therefore\ e^{\frac{1}{2}x^2}=\dfrac{1+\sqrt{5}}{2}$ {as $e^x>0$}

$\therefore\ \frac{1}{2}x^2=\ln\left(\dfrac{1+\sqrt{5}}{2}\right)$

$\therefore\ x=\pm\sqrt{2\ln\left(\dfrac{1+\sqrt{5}}{2}\right)}$

$\therefore\ x\approx 0.981$ $\{0\leqslant x\leqslant 1.5\}$

8 **a** f will not be defined when $\left|4x-x^3\right|=0$

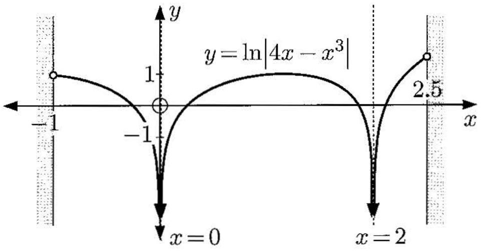
$\therefore\ (4-x)^2x=0$

$\therefore\ (2-x)(2+x)x=0$

$\therefore\ x=2,\ 0,\ -2$

But -2 is not in the given domain, so f is not defined at $x=0$ or 2. So, h and k are 0 and 2.

b



$\therefore$ there are 4 zeros

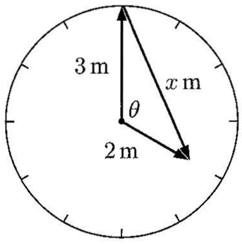
c The zeros of $f(x)$ occur if $\left|4x-x^3\right|=1$

$4x-x^3=1$ if $x\approx-1.861,\ -0.254$, or 2.115

$4x-x^3=-1$ if $x\approx-2.115$ or 1.861

$\therefore$ for $-1<x<2.5$ zeros occur at $x\approx-0.254,\ 0.254,\ 1.861,\ 2.115$

9



Hour hand moves at 1 rev/12 hours $=\dfrac{2\pi^c}{12\text{ h}}=\frac{\pi}{6}^c\text{ h}^{-1}$.

Minute hand moves at 1 rev/1 hour $=\dfrac{2\pi^c}{1\text{ h}}=2\pi^c\text{ h}^{-1}$.

$\therefore \quad \dfrac{d\theta}{dt} = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}^c \text{ h}^{-1}$

Now by the cosine rule

$$x^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos\theta$$

$$\therefore \quad x^2 = 13 - 12\cos\theta$$

$$\therefore \quad 2x\,\frac{dx}{dt} = -12(-\sin\theta)\,\frac{d\theta}{dt}$$

$$\therefore \quad x\,\frac{dx}{dt} = 6\sin\theta\,\frac{d\theta}{dt}$$

Particular case: At 4 pm, $\theta = \frac{2\pi}{3}$

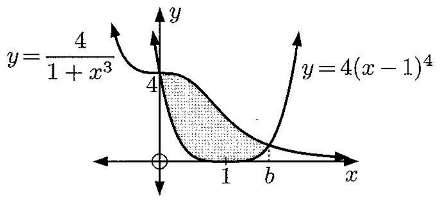
and $x^2 = 13 - 12(-\frac{1}{2}) = 19$

$$\therefore \quad \sqrt{19}\,\frac{dx}{dt} = 6 \times \tfrac{\sqrt{3}}{2} \times \left(-\tfrac{11\pi}{6}\right)$$

$$\therefore \quad \frac{dx}{dt} = -\frac{11\pi\sqrt{3}}{2\sqrt{19}} \text{ m h}^{-1} \approx -6.87 \text{ m h}^{-1}$$

$\therefore$ the distance is decreasing at 6.87 m h^{-1}.

10



Using technology, $b \approx 1.653$

$$\text{Area} \approx \int_0^{1.653} \left(\frac{4}{1+x^3} - 4(x-1)^4\right) dx \approx 3.27 \text{ units}^2$$

11 **a** The yield grows like compound interest at 3% per annum. We let the yield at end of n years be

$$Y(n) = 200\left(1 + \tfrac{3}{100}\right)^{n-1}.$$

To double the yield we need n such that

$$400 = 200\left(1 + \tfrac{3}{100}\right)^{n-1}$$

$$\therefore \quad 2 = \left(1 + \tfrac{3}{100}\right)^{n-1}$$

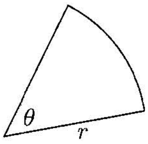
$$\therefore \quad (n-1)\ln\left(1 + \tfrac{3}{100}\right) = \ln 2$$

$$\therefore \quad n = \frac{\ln 2}{\ln(1.03)} + 1 \approx 24.45$$

So, it takes 24 years for the yield to double.

b The total yield at the end of the 8th year will be 1778 tonnes.

12



Area = 15

$$\therefore \quad \tfrac{1}{2}r^2\theta = 15$$

$$\therefore \quad r^2\theta = 30 \quad .... (1)$$

Perimeter = 16

$$\therefore \quad 2r + r\theta = 16 \quad .... (2)$$

$$\therefore \quad 2r + r\left(\frac{30}{r^2}\right) = 16$$

$$\therefore \quad 2r + \frac{30}{r} = 16$$

$$\therefore \quad 2r^2 + 30 = 16r$$

$$\therefore \quad r^2 - 8r + 15 = 0$$

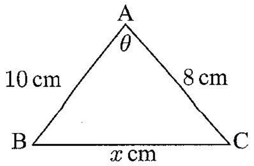
$\therefore \quad (r-3)(r-5) = 0$ and so $r = 3$ or 5.

When $r = 3$, $\theta = \dfrac{30}{3^2} = \frac{30}{9} = \frac{10}{3}$.

When $r = 5$, $\theta = \dfrac{30}{5^2} = \frac{30}{25} = \frac{6}{5}$.

So, $r = 3$, $\theta = 3\frac{1}{3}^c$ or $r = 5$, $\theta = 1.2^c$.

13



Area $= \frac{1}{2} \times 10 \times 8 \times \sin\theta = 23$

$\therefore \quad 40\sin\theta = 23$ and so $\sin\theta = \frac{23}{40}$

But $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \quad \cos^2\theta = 1 - \tfrac{23}{40} = \tfrac{17}{40}$$

$$\therefore \quad \cos\theta = \pm\sqrt{\tfrac{17}{40}}$$

Now $x^2 = 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos\theta$

$$\therefore \quad x^2 = 164 - 160\left(\pm\sqrt{\tfrac{17}{40}}\right)$$

$\therefore \quad x \approx 7.73$ or 16.38

$\therefore \quad$ BC = 7.73 cm or 16.38 cm

14 If α and β are the roots of $x^2 - kx + 4 = 0$ then $\alpha + \beta = k$ and $\alpha\beta = 4$

a $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = k^2 - 2(4) = k^2 - 8$

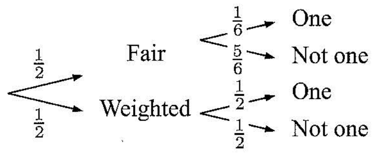
b If the roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$, then

$$\text{sum of roots} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{k}{4}$$

$$\text{product of roots} = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{4}$$

Equation could be $a\left[x^2 - \dfrac{k}{4}x + \frac{1}{4}\right] = 0$, $a \neq 0$.

15 Consider this tree diagram:



a P(One) = P(Fair ∩ One or Weighted ∩ One)

$$= \tfrac{1}{2} \times \tfrac{1}{6} + \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{3}$$

b $$\text{P(Weighted | One)} = \frac{\text{P(Weighted} \cap \text{One)}}{\text{P(One)}} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{3}} = \frac{3}{4}$$

16 $(1+x)^{2n} = \displaystyle\sum_{k=0}^{2n} \binom{2n}{k} x^k$ and

$$(1+15x^2)^n = \sum_{l=0}^{n} \binom{n}{l} (15x^2)^l$$

The terms involving x^2 occur when $k = 2$ and $l = 1$.

If the coefficients of x^2 are equal, then

$$\binom{2n}{2} = \binom{n}{1}(15)^1$$

$$\therefore \quad \frac{(2n)(2n-1)}{2!} = n \times 15$$

$$\therefore \quad 2n - 1 = 15 \quad \{\text{as } n \neq 0\}$$

$$\therefore \quad n = 8$$

17 Let X be the net weight of a jar, then $X \sim \text{N}(475, 7.5^2)$.

$\therefore \quad \text{P}(X < 460) \approx 0.0228$

So, about 2.28% of the jars have weight less than 460 g.

18 A normal to the first plane is

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-2+1)\mathbf{i} - (4-1)\mathbf{j} + (-2+1)\mathbf{k}$$

$$= -\mathbf{i} - 3\mathbf{j} - \mathbf{k}$$

A normal to the second plane is

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{k}$$

$$= (-1-0)\mathbf{i} - (-1-2)\mathbf{j} + (0-2)\mathbf{k}$$

$$= -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

$$\therefore \quad \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$$

$$= \frac{|1-9+2|}{\sqrt{1+9+1}\sqrt{1+9+4}}$$

$$= \frac{6}{\sqrt{11}\sqrt{14}} \quad \text{and so} \quad \theta \approx 61.1^\circ$$

19 If $Y \sim \text{Po}(m)$, then $\text{E}(Y) = \text{Var}(Y) = m$

$$\therefore \quad m^2 = 2m + 3$$

$$\therefore \quad m^2 - 2m - 3 = 0$$

$$\therefore \quad (m-3)(m+1) = 0$$

$$\therefore \quad m = 3 \quad \text{or} \quad -1$$

But $m > 0$, so $m = 3$.

$$\text{P}(Y \geqslant 3) = 1 - \text{P}(Y \leqslant 2)$$

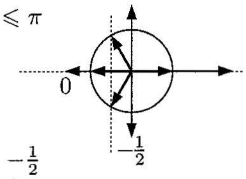
$$\approx 1 - 0.423$$

$$\approx 0.577$$

20 **a** $f(x) = \sin 2x + \sin 4x, \quad 0 \leqslant x \leqslant \pi$

$$= \sin 2x + 2\sin 2x \cos 2x$$

$$= \sin 2x(1 + 2\cos 2x)$$



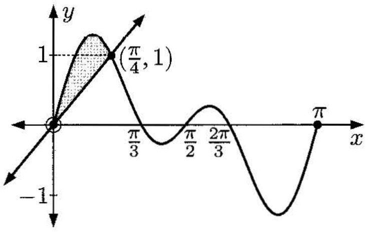
$\therefore \quad f(x) = 0$ when

$\sin 2x = 0 \quad$ or $\quad \cos 2x = -\frac{1}{2}$

$\therefore \quad 2x = k\pi \quad$ or $\quad 2x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$\therefore \quad x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

Using technology, the graph is



b When $x = 0$, $y = \frac{4}{\pi}(0) = 0$ and $f(0) = 0 + 0 = 0$.

When $x = \frac{\pi}{4}$, $y = \frac{4}{\pi}\left(\frac{\pi}{4}\right) = 1$ and

$f\left(\frac{\pi}{4}\right) = \sin(\frac{\pi}{2}) + \sin(\pi) = 1 + 0 = 1$

$\therefore \quad y = \frac{4}{\pi}x$ meets $y = f(x)$ at the origin and at $(\frac{\pi}{4}, 1)$.

c Shaded area $= \displaystyle\int_0^{\frac{\pi}{4}} \left(f(x) - \tfrac{4}{\pi}x\right) dx$

$$= \int_0^{\frac{\pi}{4}} \left(\sin 2x + \sin 4x - \tfrac{4}{\pi}x\right) dx$$

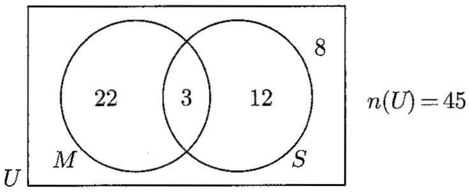
$$= \left[-\tfrac{1}{2}\cos 2x - \tfrac{1}{4}\cos 4x - \tfrac{4}{\pi}\frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$$

$$= -\tfrac{1}{2}(0) - \tfrac{1}{4}(-1) - \tfrac{2}{\pi}\left(\tfrac{\pi^2}{16}\right) + \tfrac{1}{2}(1) + \tfrac{1}{4}(1) + 0$$

$$= \tfrac{1}{4} - \tfrac{\pi}{8} + \tfrac{1}{2} + \tfrac{1}{4}$$

$$= \left(1 - \tfrac{\pi}{8}\right) \text{ units}^2$$

21 Let M = students with American passport
S = students with Australian passport



Total number of students is $45 = 8 + 25 + 15 - n(M \cap S)$

$$\therefore \quad n(M \cap S) = 3$$

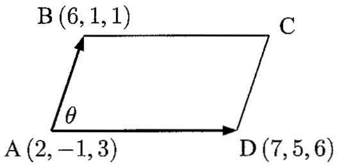
So there are 3 students who hold both American and Australian passports.

a P(student holds both passports) $= \frac{3}{45} = \frac{1}{15}$

b P(student has neither passport) $= \frac{8}{45}$

c P(student has exactly one passport) $= \frac{34}{45}$

22



$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}, \quad \overrightarrow{AD} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & -2 \\ 5 & 6 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 6 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -2 \\ 5 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix}\mathbf{k}$$

$$= 18\mathbf{i} - 22\mathbf{j} + 14\mathbf{k}$$

Area of parallelogram $= |\,\overrightarrow{AB} \times \overrightarrow{AD}\,|$

$$= \sqrt{18^2 + (-22)^2 + 14^2} \text{ units}^2$$

$$= \sqrt{1004} \text{ units}^2$$

$$\approx 31.7 \text{ units}^2$$

23 Let X be the lengths of the fish, so $X \sim \text{N}(m, 0.12^2)$.

$$\text{P}(X \geqslant 13) = 0.2$$

$$\therefore \quad \text{P}\left(\frac{X-m}{0.12} \geqslant \frac{13-m}{0.12}\right) = 0.2$$

$$\therefore \quad \text{P}\left(Z \geqslant \frac{13-m}{0.12}\right) = 0.2$$

$$\therefore \quad \text{P}\left(Z \leqslant \frac{13-m}{0.12}\right) = 0.8$$

$$\therefore \quad \frac{13-m}{0.12} \approx 0.842$$

$$\therefore \quad m \approx 13 - 0.12 \times 0.842$$

$$\therefore \quad m \approx 12.9$$

The mean length of fish is about 12.9 cm.

24 **a** Mean score $\mu = \frac{109}{25} = 4.36$

b Variance $\sigma^2 = \dfrac{\sum x_r^2}{n} - \mu^2$

$$= \frac{579}{25} - 4.36^2$$
$$\approx 4.15$$

25 **a** L_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 3+3\lambda \\ 1+2\lambda \end{pmatrix}$

L_2: $\mathbf{r} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 3-4\mu \\ \frac{3}{2}+\frac{3}{2}\mu \\ -1+2\mu \end{pmatrix}$

b If the lines intersect, then for some λ and μ

$$1+2\lambda = 3-4\mu \quad .... (1)$$
$$3+3\lambda = \tfrac{3}{2}+\tfrac{3}{2}\mu \quad .... (2)$$
$$1+2\lambda = -1+2\mu \quad .... (3)$$

Using (1) and (3) we have $2\lambda + 4\mu = 2$ .... (1)
$2\lambda - 2\mu = -2$ .... (3)

From (1) – (3), $6\mu = 4$ or $\mu = \frac{2}{3}$

From (1), $\lambda = 1 - 2\mu = 1 - 2(\frac{2}{3}) = -\frac{1}{3}$

But, for (2), $3 + 3\lambda = 3 + 3(-\frac{1}{3}) = 2$

and $\frac{3}{2} + \frac{3}{2}\mu = \frac{3}{2} + \frac{3}{2}(\frac{2}{3}) = \frac{5}{2}$

So, equation (2) is not satisfied.

Hence, the lines do not intersect.

Since $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix}$ are not parallel, the lines are not parallel.

c A normal to the plane is $\begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix}$.

Since the point $(1, 3, 1)$ lies on the line L_1, the plane

$$\begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore$ $-4x + \frac{3}{2}y + 2z = \frac{5}{2}$ which is $8x - 3y - 4z = 5$ is perpendicular to L_2 and intersects L_1.

d A vector perpendicular to both lines is

$$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -4 & \frac{3}{2} & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 \\ \frac{3}{2} & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -4 & \frac{3}{2} \end{vmatrix} \mathbf{k}$$

$$= (6-3)\mathbf{i} - (4+8)\mathbf{j} + (3+12)\mathbf{k}$$
$$= 3\mathbf{i} - 12\mathbf{j} + 15\mathbf{k}$$
$$= 3(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$$

e Let X be a typical point on line L_1 and Y on line L_2.

So, $\overrightarrow{\text{XY}} = \left(\begin{pmatrix} 3 \\ \frac{3}{2} \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ \frac{3}{2} \\ 2 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}\right)\right)$

$$= \begin{pmatrix} 2-4\mu-2\lambda \\ -\frac{3}{2}+\frac{3}{2}\mu-3\lambda \\ -2+2\mu-2\lambda \end{pmatrix}$$

From **d**, this is perpendicular to both lines if there is a k such that $\begin{pmatrix} 2-4\mu-2\lambda \\ -\frac{3}{2}+\frac{3}{2}\mu-3\lambda \\ -2+2\mu-2\lambda \end{pmatrix} = k \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

This gives the 3 equations $4\mu + 2\lambda + k = 2$

$$\tfrac{3}{2}\mu - 3\lambda + 4k = \tfrac{3}{2}$$
$$2\mu - 2\lambda - 5k = 2$$

Using technology, this system has the solution $k = -\frac{1}{21}$.

So, the distance between the lines is $\left| -\frac{1}{21} \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} \right|$

$$= \tfrac{\sqrt{42}}{21} \text{ units}$$

SOLUTIONS TO EXAMINATION PRACTICE SET 7

1 **a** Let $z = x + iy$, $x, y \in \mathbb{R}$

$$\therefore \quad (x+iy)^2 = 2(x-iy)$$
$$\therefore \quad x^2 - y^2 + 2xyi = 2x - 2yi$$
$$\therefore \quad x^2 - y^2 = 2x \quad \text{and} \quad 2xy = -2y$$
$$\therefore \quad 2y(x+1) = 0$$
$$\therefore \quad y = 0 \quad \text{or} \quad x = -1$$

As z is non-real, $y \neq 0$ and so $x = -1$

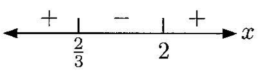
Hence, $1 - y^2 = -2$

$$\therefore \quad y^2 = 3$$
$$\therefore \quad y = \pm\sqrt{3}$$

Thus, $z = -1 + i\sqrt{3}$ or $-1 - i\sqrt{3}$

b

$$|z| < 2\,|z - 1 - i|$$
$$\therefore \quad \sqrt{x^2+4} < 2\,|x + 2i - 1 - i|$$
$$\therefore \quad \sqrt{x^2+4} < 2\sqrt{(x-1)^2+1}$$
$$\therefore \quad x^2 + 4 < 4(x^2 - 2x + 2)$$
$$\therefore \quad x^2 + 4 < 4x^2 - 8x + 8$$
$$\therefore \quad 3x^2 - 8x + 4 > 0$$
$$\therefore \quad (3x-2)(x-2) > 0$$
$$\therefore \quad x \in \left]-\infty, \tfrac{2}{3}\right[\cup \left]2, \infty\right[$$



2

$$\log_6(x+3) = 1 - \log_6(x-2)$$
$$\therefore \quad \log_6(x+3) + \log_6(x-2) = 1$$
$$\therefore \quad \log_6[(x+3)(x-2)] = 1$$
$$\therefore \quad (x+3)(x-2) = 6^1$$
$$\therefore \quad x^2 + x - 6 = 6$$
$$\therefore \quad x^2 + x - 12 = 0$$
$$\therefore \quad (x+4)(x-3) = 0$$
$$\therefore \quad x = -4 \text{ or } 3$$

But $x + 3 > 0$ and $x - 2 > 0$

$$\therefore \quad x > 2$$

Hence, $x = 3$

3

$$y = xe^{\;x^2}$$
$$\therefore \quad \frac{dy}{dx} = 1e^{-x^2} + xe^{-x^2}(-2x)$$
$$= e^{-x^2}(1 - 2x^2)$$
$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } x^2 = \tfrac{1}{2} \quad \therefore \quad x = \pm\tfrac{1}{\sqrt{2}}$$

When $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$.

When $x = -\sqrt{2}$, $y = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$.

$\therefore$ the stationary points are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}})$.

4 **a** We have a geometric series where

$$u_1 - u_2 = 9 \quad \text{and} \quad S = \frac{u_1}{1-r} = 81$$

Thus $u_1 - u_1 r = 9$ and $u_1 = 81(1-r)$

$$\therefore \quad u_1(1-r) = 9 \quad \text{and} \quad u_1 = 81(1-r)$$

$$\therefore \quad \frac{9}{1-r} = 81(1-r)$$

$$\therefore \quad (1-r)^2 = \tfrac{1}{9}$$

$$\therefore \quad 1-r = \pm\tfrac{1}{3} \text{ and so } r = \tfrac{4}{3} \text{ or } \tfrac{2}{3}$$

But $-1 < r < 1$, so $r = \frac{2}{3}$.

b $u_1 = 81(1-r) = 81 \times \frac{1}{3} = 27$

So, the first term is 27.

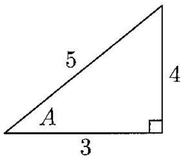
5 **a**

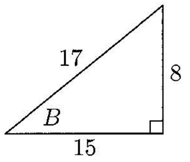
$$\begin{aligned} &g(x) \\ &= f(f(x)) \\ &= f\left(\frac{x}{x-2}\right) \\ &= \frac{\frac{x}{x-2}}{\frac{x}{x-2} - 2} \\ &= \frac{x}{x-2(x-2)} \\ &= \frac{x}{-x+4} \end{aligned}$$

b

$$\begin{aligned} &g(g(2)) \\ &= g\left(\frac{2}{-2+4}\right) \\ &= g(1) \\ &= \frac{1}{-1+4} \\ &= \tfrac{1}{3} \end{aligned}$$

6





$$\therefore \quad \tan A = \tfrac{4}{3} \quad \text{and} \quad \tan B = \tfrac{8}{15}$$

$$\begin{aligned} \text{So,} \quad \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{4}{3} + \frac{8}{15}}{1 - \left(\frac{4}{3}\right)\left(\frac{8}{15}\right)} \times \frac{45}{45} \\ &= \frac{60+24}{45-32} \\ &= \tfrac{84}{13} \end{aligned}$$

7

$$\sqrt{2}\sin x = \tan x, \quad 0 \leqslant x \leqslant 2\pi$$

$$\therefore \quad \sqrt{2}\sin x = \frac{\sin x}{\cos x}$$

$$\therefore \quad \sqrt{2}\sin x \cos x = \sin x$$

$$\therefore \quad \sqrt{2}\sin x \cos x - \sin x = 0$$

$$\therefore \quad \sin x(\sqrt{2}\cos x - 1) = 0$$

$$\therefore \quad \sin x = 0 \quad \text{or} \quad \cos x = \tfrac{1}{\sqrt{2}}$$

$$\therefore \quad x = 0,\ \tfrac{\pi}{4},\ \pi,\ \tfrac{7\pi}{4},\ 2\pi$$

8 **a**

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= (2+3)\mathbf{i} - (2+0)\mathbf{j} + (1-0)\mathbf{k} \\ &= \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

b $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.

$|\mathbf{a} \times \mathbf{b}| = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}$

A vector perpendicular to **a** and **b** of length 5 units is

$$\tfrac{5}{\sqrt{30}}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{25}{\sqrt{30}} \\ -\frac{10}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{pmatrix}.$$

9 $f(x) = \dfrac{ax+b}{x^2-5x+7}$

We are given $f(3) = 5$ so $\dfrac{3a+b}{9-15+7} = 5$

$$\therefore \quad 3a + b = 5 \quad .... (1)$$

$$\begin{aligned} \text{Now} \quad f'(x) &= \frac{a(x^2-5x+7) - (ax+b)(2x-5)}{(x^2-5x+7)^2} \\ &= \frac{ax^2 - 5ax + 7a - 2ax^2 - 2bx + 5ax + 5b}{(x^2-5x+7)^2} \\ &= \frac{-ax^2 - 2bx + 7a + 5b}{(x^2-5x+7)^2} \end{aligned}$$

But $f'(3) = 0$ so $-9a - 6b + 7a + 5b = 0$

$$\therefore \quad -2a - b = 0$$

$$\therefore \quad b = -2a \quad .... (2)$$

Substituting (2) into (1), $3a - 2a = 5$

$$\therefore \quad a = 5 \text{ and so } b = -10$$

10 **a** Displacement $s = A\cos 2t + B\sin 2t$ metres

$\therefore$ velocity $v = -2A\sin 2t + 2B\cos 2t \text{ m s}^{-1}$

$$\begin{aligned} \therefore \text{ acceleration } a &= -4A\cos 2t - 4B\sin 2t \text{ m s}^{-2} \\ &= -4(A\cos 2t + B\sin 2t) \text{ m s}^{-2} \\ &= -4s \text{ m s}^{-2} \end{aligned}$$

b At time $t = 0$, $s = 5$ m and $v = 0 \text{ m s}^{-1}$

$$\therefore \quad A\cos 0 + B\sin 0 = 5$$

$$\therefore \quad A = 5$$

and $-2A\sin 0 + 2B\cos 0 = 0$

$$\therefore \quad B = 0$$

c Using **b**, $s = 5\cos 2t$ m

and $v = -10\sin 2t \text{ m s}^{-1}$

The particle has displacement 3 m when $5\cos 2t = 3$

$$\therefore \quad \cos 2t = \tfrac{3}{5}$$

Since $t > 0$, $\sin 2t > 0$,

so $\sin 2t = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$

$\therefore$ the velocity is $-10 \times \frac{4}{5} = -8 \text{ m s}^{-1}$.

11 $y^2 + 2y - 3x = 0$ meets $2x - y - 1 = 0$ where

$$y^2 + 2y - 3\left(\frac{y+1}{2}\right) = 0$$

$$\therefore \quad 2y^2 + 4y - 3y - 3 = 0$$

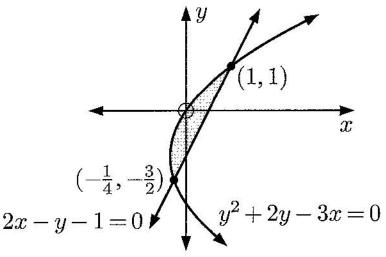
$$\therefore \quad 2y^2 + y - 3 = 0$$

$$\therefore \quad (2y+3)(y-1) = 0$$

$$\therefore \quad y = -\tfrac{3}{2} \text{ or } 1$$

When $y = -\frac{3}{2}$, $x = \dfrac{-\frac{3}{2}+1}{2} = -\frac{1}{4}$

When $y = 1$, $x = \dfrac{1+1}{2} = 1$

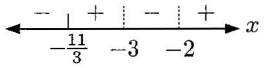


$$\begin{aligned}\text{Area} &= \int_{-\frac{3}{2}}^{1}\left(\left(\frac{y+1}{2}\right)-\left(\frac{y^2+2y}{3}\right)\right)dy\\ &= \int_{-\frac{3}{2}}^{1}\left(\tfrac{1}{2}y+\tfrac{1}{2}-\tfrac{1}{3}y^2-\tfrac{2}{3}y\right)dy\\ &= \int_{-\frac{3}{2}}^{1}\left(\tfrac{1}{2}-\tfrac{1}{3}y^2-\tfrac{1}{6}y\right)dy\\ &= \left[\tfrac{1}{2}y-\tfrac{1}{9}y^3-\tfrac{1}{12}y^2\right]_{-\frac{3}{2}}^{1}\\ &= \left(\tfrac{1}{2}-\tfrac{1}{9}-\tfrac{1}{12}\right)-\left(-\tfrac{3}{4}+\tfrac{3}{8}-\tfrac{3}{16}\right)\\ &= \tfrac{125}{144}\text{ units}^2\end{aligned}$$

12

$$\begin{aligned}\frac{5}{x+2} &\geqslant \frac{2}{x+3}\\ \therefore\ \frac{5}{x+2}-\frac{2}{x+3} &\geqslant 0\\ \therefore\ \frac{5x+15}{(x+2)(x+3)}-\frac{2(x+2)}{(x+2)(x+3)} &\geqslant 0\\ \therefore\ \frac{5x+15-2x-4}{(x+2)(x+3)} &\geqslant 0\\ \therefore\ \frac{3x+11}{(x+2)(x+3)} &\geqslant 0\end{aligned}$$

Sign diagram:



Inequality is satisfied if $-\frac{11}{3} \leqslant x < -3$ or $x > -2$.

13 $$\begin{aligned}\tan 75^\circ &= \tan(45^\circ+30^\circ)\\ &= \frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}\\ &= \frac{1+\frac{1}{\sqrt{3}}}{1-(1)\frac{1}{\sqrt{3}}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{4+2\sqrt{3}}{2}\\ &= 2+\sqrt{3}\end{aligned}$$

14 **a** The two lines meet if $\begin{pmatrix}3+4t\\4+t\\1\end{pmatrix}=\begin{pmatrix}-1+12\lambda\\7+6\lambda\\5+3\lambda\end{pmatrix}$

$$\begin{aligned}\therefore\ 3+4t &= -1+12\lambda \quad \text{.... (1)}\\ 4+t &= 7+6\lambda \quad \text{.... (2)}\\ 1 &= 5+3\lambda \quad \text{.... (3)}\end{aligned}$$

From (3), $\lambda = -\frac{4}{3}$

From (2), $4+t = 7+6\left(-\frac{4}{3}\right) = 7-8$

$\therefore\ t = -5$

These values check in equation (1).

So, the point $(-17, -1, 1)$ lies on both lines.

b A normal to both lines is

$$\begin{aligned}\mathbf{n} = \begin{pmatrix}4\\1\\0\end{pmatrix}\times\begin{pmatrix}12\\6\\3\end{pmatrix} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & 0\\ 12 & 6 & 3\end{vmatrix}\\ &= (3-0)\mathbf{i}-(12-0)\mathbf{j}+(24-12)\mathbf{k}\\ &= 3\mathbf{i}-12\mathbf{j}+12\mathbf{k}\\ &= 3(\mathbf{i}-4\mathbf{j}+4\mathbf{k})\end{aligned}$$

Since $(-17, -1, 1)$ lies on this plane, an equation is

$$\begin{aligned}x-4y+4z &= -17-4(-1)+4(1)\\ \therefore\ x-4y+4z &= -9\end{aligned}$$

15 We integrate by parts with $u = \arctan x$ $\quad v' = x$

$u' = \dfrac{1}{1+x^2}$ $\quad v = \dfrac{x^2}{2}$

$$\begin{aligned}\therefore\ &\int x\arctan x\,dx\\ &= \frac{x^2}{2}\arctan x-\int\frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx\\ &= \frac{x^2}{2}\arctan x-\tfrac{1}{2}\int\frac{x^2}{1+x^2}\,dx\\ &= \frac{x^2}{2}\arctan x-\tfrac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx\\ &= \frac{x^2}{2}\arctan x-\tfrac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx\\ &= \frac{x^2}{2}\arctan x-\tfrac{1}{2}\left(x-\arctan x\right)+c\\ &= \frac{x^2}{2}\arctan x-\tfrac{1}{2}x+\tfrac{1}{2}\arctan x+c\end{aligned}$$

16

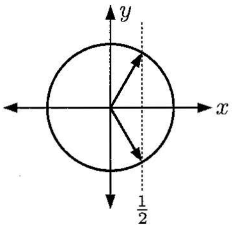
$$\begin{aligned}\log\left(x^2y^3\right) &= a\\ \therefore\ \log(x^2)+\log(y^3) &= a\\ \therefore\ 2\log x+3\log y &= a \quad \text{.... (1)}\\ \log\frac{x}{y} &= b\\ \therefore\ \log x-\log y &= b \quad \text{.... (2)}\end{aligned}$$

$(1)+3\times(2)$ gives $5\log x = a+3b$ $\quad\therefore\ \log x = \dfrac{a+3b}{5}$

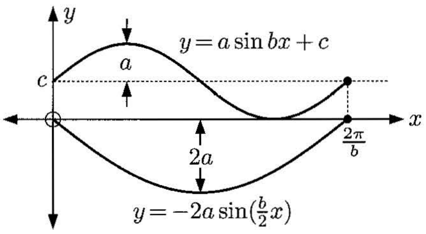
$(1)-2\times(2)$ gives $5\log y = a-2b$ $\quad\therefore\ \log y = \dfrac{a-2b}{5}$

17

$$\begin{aligned}2\cos\theta+2\sec\theta &= 5,\quad 0\leqslant\theta\leqslant\pi\\ \therefore\ 2\cos\theta+\frac{2}{\cos\theta}-5 &= 0\\ \therefore\ 2\cos^2\theta-5\cos\theta+2 &= 0\\ \therefore\ (2\cos\theta-1)(\cos\theta-2) &= 0\\ \therefore\ \cos\theta &= \tfrac{1}{2}\text{ or }2\\ \therefore\ \cos\theta &= \tfrac{1}{2}\quad\{\text{as } -1\leqslant\cos\theta\leqslant 1\}\\ \therefore\ \theta &= \tfrac{\pi}{3}\quad\{0\leqslant\theta\leqslant\pi\}\end{aligned}$$



18



$y = a\sin bx+c$ has y-intercept c, period $\frac{2\pi}{b}$, and amplitude a.

$y = -2a\sin\left(\frac{b}{2}\right)x$ has y-intercept 0, period $\dfrac{2\pi}{\frac{b}{2}} = \frac{4\pi}{b}$, and amplitude $2a$.

19

$$y = Ae^{kt}$$

$\therefore\ \dfrac{dy}{dt} = Ake^{kt}$ and $\dfrac{d^2y}{dt^2} = Ak^2e^{kt}$

But $\dfrac{d^2y}{dt^2}+3\dfrac{dy}{dt}+2y = 0$

$$\therefore \quad Ak^2e^{kt} + 3Ake^{kt} + 2Ae^{kt} = 0$$
$$\therefore \quad Ae^{kt}(k^2+3k+2) = 0$$
$$\therefore \quad k^2+3k+2 = 0 \quad \{\text{since } e^{kt} > 0\}$$
$$\therefore \quad (k+1)(k+2) = 0 \text{ and so } k = -1 \text{ or } -2$$

20 If $y^2 = 3 - xy$ then $2y\,\frac{dy}{dx} = 0 - \left(1y + x\,\frac{dy}{dx}\right)$

$$\therefore \quad 2y\,\frac{dy}{dx} = -y - x\,\frac{dy}{dx}$$
$$\therefore \quad (x+2y)\,\frac{dy}{dx} = -y$$
$$\therefore \quad \frac{dy}{dx} = \frac{-y}{x+2y}$$

If a tangent has gradient $-\frac{3}{4}$, then $\frac{-y}{x+2y} = -\frac{3}{4}$

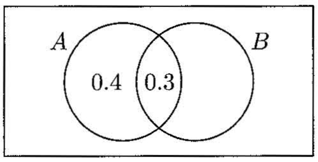
$$\therefore \quad 4y = 3(x+2y)$$
$$\therefore \quad 4y = 3x + 6y$$
$$\therefore \quad -2y = 3x$$
$$\therefore \quad y = -\frac{3x}{2}$$

Substituting into $y^2 = 3 - xy$ gives

$$\frac{9x^2}{4} = 3 - x\left(\frac{-3x}{2}\right)$$
$$\therefore \quad \frac{9x^2}{4} = 3 + \frac{3x^2}{2}$$
$$\therefore \quad 9x^2 = 12 + 6x^2$$
$$\therefore \quad 3x^2 = 12$$
$$\therefore \quad x^2 = 4 \text{ and so } x = \pm 2$$

$\therefore$ the points are $(2, -3)$ and $(-2, 3)$.

21



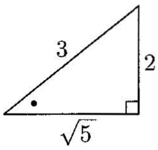
$$\text{P}(A) = \text{P}(A \cap B) + \text{P}(A \cap B') \quad \{\text{disjoint sets}\}$$
$$= 0.7$$

and $\text{P}(A \cap B) = \text{P}(A)\text{P}(B)$ $\{A \text{ and } B \text{ are independent}\}$

$$\therefore \quad \text{P}(B) = \frac{0.3}{0.7} = \frac{3}{7}$$
$$\therefore \quad \text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$
$$= \tfrac{7}{10} + \tfrac{3}{7} - \tfrac{3}{10}$$
$$= \tfrac{29}{35}$$

22 $\sin A = -\frac{2}{3}$, $0 < A < \frac{3\pi}{2}$ so A is in quadrant 3.

$$\tan A = \tfrac{2}{\sqrt{5}}$$



Now $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$$\therefore \quad \tan 2A = \frac{\frac{4}{\sqrt{5}}}{1 - \frac{4}{5}}$$
$$= \frac{\frac{4}{\sqrt{5}}}{\frac{1}{5}} = \tfrac{4}{\sqrt{5}} \times 5 = 4\sqrt{5}$$

23 **a**

$$z = re^{i\theta}$$
$$\therefore \quad z^n = (re^{i\theta})^n$$
$$= r^n e^{in\theta}$$
$$= r^n\,(\cos(n\theta) + i\sin(n\theta))$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

b **i** $1+i$ has $|1+i| = \sqrt{1+1} = \sqrt{2}$

$$\therefore \quad 1+i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)$$
$$= \sqrt{2}\,\text{cis}\left(\tfrac{\pi}{4}\right)$$

$1-i$ has $|1-i| = \sqrt{1+1} = \sqrt{2}$

$$\therefore \quad 1-i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)$$
$$= \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)$$

ii $(1+i)^n + (1-i)^n$

$$= \left[\sqrt{2}\,\text{cis}\left(\tfrac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)\right]^n$$
$$= 2^{\frac{n}{2}}\text{cis}\left(\tfrac{n\pi}{4}\right) + 2^{\frac{n}{2}}\text{cis}\left(-\tfrac{n\pi}{4}\right) \quad \{\text{De Moivre}\}$$
$$= 2^{\frac{n}{2}}\left(\cos\tfrac{n\pi}{4} + i\sin\tfrac{n\pi}{4} + \cos\tfrac{n\pi}{4} - i\sin\tfrac{n\pi}{4}\right)$$
$$= 2^{\frac{n}{2}} \times 2\cos\left(\tfrac{n\pi}{4}\right)$$
$$= 2^{\frac{n}{2}+1}\cos\left(\tfrac{n\pi}{4}\right)$$

iii If $(1+i)^n + (1-i)^n = 64$

then $2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} = 2^6$

$$\therefore \quad \cos\tfrac{n\pi}{4} = 2^{6-1-\frac{n}{2}}$$
$$\therefore \quad \cos\tfrac{n\pi}{4} = 2^{5-\frac{n}{2}}$$

But the LHS $= \pm\frac{1}{\sqrt{2}}, 0, \pm 1$

Also, $2^{5-\frac{n}{2}} > 0$ for all n, so $\cos\frac{n\pi}{4} > 0$

So, the only values of n where the equation might be true is when

$$2^{5-\frac{n}{2}} = 2^{-\frac{1}{2}} \text{ or } 2^0$$
$$\therefore \quad 5 - \frac{n}{2} = -\tfrac{1}{2} \text{ or } 0$$
$$\therefore \quad \frac{n}{2} = \tfrac{11}{2} \text{ or } 5$$
$$\therefore \quad n = 10 \text{ or } 11$$

If $n = 10$, $\cos\frac{5\pi}{2} = 2^0$

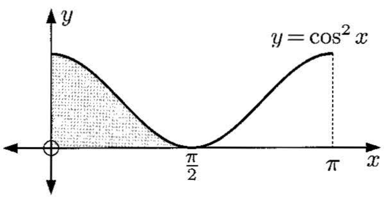
$$\therefore \quad \cos\tfrac{\pi}{2} = 1$$
$$\therefore \quad 0 = 1 \quad \text{false}$$

If $n = 11$, $\cos\frac{11\pi}{4} = 2^{-\frac{1}{2}}$

$$\therefore \quad \cos\tfrac{3\pi}{4} = \tfrac{1}{\sqrt{2}}$$
$$\therefore \quad -\tfrac{1}{\sqrt{2}} = \tfrac{1}{\sqrt{2}} \quad \text{false}$$

So, no integer solutions exist.

24 **a**



$$\text{Area} = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)\,dx$$
$$= \left[\frac{x}{2} + \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi - (0+0)$$
$$= \tfrac{\pi}{4} \text{ units}^2$$

b $\int \cos^n x\,dx = \int \cos^{n-1} x\,\cos x\,dx$

We integrate by parts with

$u = \cos^{n-1} x \qquad v' = \cos x$

$u' = (n-1)\cos^{n-2} x(-\sin x) \qquad v = \sin x$

$$\therefore \int \cos^n x\,dx$$
$$= \cos^{n-1} x\,\sin x - \int (n-1)\cos^{n-2} x(-\sin x)(\sin x)\,dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\,dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$
$$-(n-1)\int \cos^n x\,dx$$
$$\therefore (1+n-1)\int \cos^n x\,dx$$
$$= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$
$$\therefore n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$
$$\therefore \int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

c $\text{Volume} = \pi\int_0^{\frac{\pi}{2}} y^2\,dx = \pi\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$

Now $\int \cos^4 x\,dx = \frac{1}{4}\sin x\cos^3 x + \frac{3}{4}\int \cos^2 x\,dx$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left[\tfrac{1}{4}\sin x\cos^3 x\right]_0^{\frac{\pi}{2}} + \tfrac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$
$$= \tfrac{\pi}{8}(0) - (0) + \tfrac{3}{4}\times\tfrac{\pi}{4}$$
$$= \tfrac{3\pi}{16}$$
$$\therefore \text{volume} = \pi\times\tfrac{3\pi}{16} = \tfrac{3\pi^2}{16}\text{ units}^3$$

25 **a** $\overrightarrow{\text{OM}} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

b $\overrightarrow{\text{OM}} \bullet \overrightarrow{\text{AB}} = \frac{1}{2}(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{b}-\mathbf{a})$

$$= \tfrac{1}{2}(\mathbf{a}\bullet\mathbf{b} - \mathbf{a}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{a})$$
$$= \tfrac{1}{2}(-|\mathbf{a}|^2 + |\mathbf{b}|^2)$$

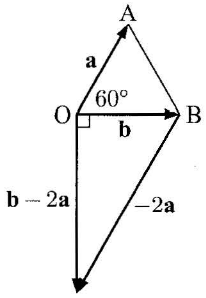
Since $\triangle$OAB is equilateral, $|\mathbf{a}|^2 = |\mathbf{b}|^2$ and $\overrightarrow{\text{OM}} \bullet \overrightarrow{\text{AB}} = 0$

$\therefore$ $\overrightarrow{\text{OM}}$ is perpendicular to $\overrightarrow{\text{AB}}$.

c $|\mathbf{b}-\mathbf{a}|$ is the length of side [AB]. Since $\triangle$OAB is equilateral, all sides are equal and $|\mathbf{b}-\mathbf{a}| = |\mathbf{a}|$.

$$\mathbf{b}\bullet(\mathbf{b}-2\mathbf{a}) = \mathbf{b}\bullet(\mathbf{b}-\mathbf{a}) - \mathbf{b}\bullet\mathbf{a}$$
$$= |\mathbf{b}|\,|\mathbf{b}-\mathbf{a}|\cos 60^\circ - |\mathbf{b}|\,|\mathbf{a}|\cos 60^\circ$$
$$= |\mathbf{a}|^2\tfrac{1}{2} - |\mathbf{a}|^2\tfrac{1}{2}$$
$$= 0$$

d



SOLUTIONS TO EXAMINATION PRACTICE SET 8

1 $u_1 = 500$, $d = 50$, arithmetic

a u_{30}
$= u_1 + 29d$
$= 500 + 29\times 50$
$= 500\times 1450$
$= 1950$ metres

b $u_1 + u_2 + u_3 + ... + u_{30}$
$= S_{30}$
$= \frac{30}{2}(2u_1 + (30-1)d)$
$= 15(1000 + 29\times 50)$
$= 36\,750$ metres

2 Using the binomial theorem, $\left(x+\dfrac{1}{x}\right)^9$ has general term

$T_{r+1} = \binom{9}{r}x^{9-r}\left(x^{-1}\right)^r = \binom{9}{r}x^{9-2r}$

a $9-2r = -1$ if $2r = 10$ or $r = 5$

$\therefore$ coefficient of $\dfrac{1}{x}$ is $\binom{9}{5} = 126$

b $9-2r = -2$ if $2r = 11$ or $r = \frac{11}{2}$

But r is an integer, so the coefficient of $\dfrac{1}{x^2}$ is 0.

3 **a** $W_t = W_0 e^{-\frac{t}{5000}}$

If $W_t = \frac{1}{2}W_0$ then $\frac{1}{2}W_0 = W_0 e^{-\frac{t}{5000}}$

$$\therefore \tfrac{1}{2} = e^{-\frac{t}{5000}}$$
$$\therefore \ln(\tfrac{1}{2}) = -\tfrac{t}{5000}$$
$$\therefore t = -5000\ln(\tfrac{1}{2}) \approx 3465\text{ years}$$

b To fall to 0.1% of its original value

$$0.001W_0 = W_0 e^{-\frac{t}{5000}}$$
$$\therefore 0.001 = e^{-\frac{t}{5000}}$$
$$\therefore \ln(0.001) = -\tfrac{t}{5000}$$
$$\therefore t = -5000\times\ln(0.001) \approx 34\,500\text{ years}$$

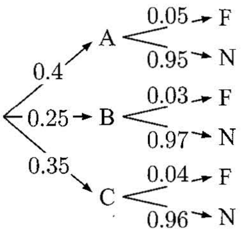
c After 1000 years $W_t = W_0 e^{-\frac{1000}{5000}} = W_0 e^{-0.2}$

Weight loss is $W_0 - W_0 e^{-0.2} = W_0(1-e^{-0.2})$

the percent weight loss is $\dfrac{W_0(1-e^{-0.2})}{W_0}\times 100\%$

$\approx 18.1\%$

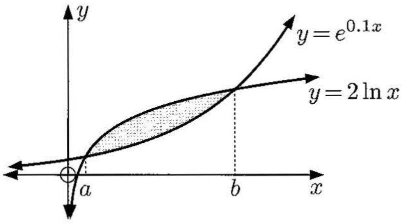
4 The situation can be represented by a tree diagram.



P(a component is faulty)

$= 0.4\times 0.05 + 0.25\times 0.03 + 0.35\times 0.04 = 0.041$

5



Using technology, $a \approx 1.821$, $b \approx 17.43$ and area ≈ 21.1 units^2.

6 **a** $\overrightarrow{\text{AB}} = \begin{pmatrix}-2\\-1\\6\end{pmatrix}$, $\overrightarrow{\text{AC}} = \begin{pmatrix}5\\2\\2\end{pmatrix}$

and $\overrightarrow{\text{AB}} \bullet \overrightarrow{\text{AC}} = \begin{pmatrix}-2\\-1\\6\end{pmatrix} \bullet \begin{pmatrix}5\\2\\2\end{pmatrix} = -10 - 2 + 12 = 0$

So, $\overrightarrow{\text{AB}}$ is perpendicular to $\overrightarrow{\text{AC}}$.

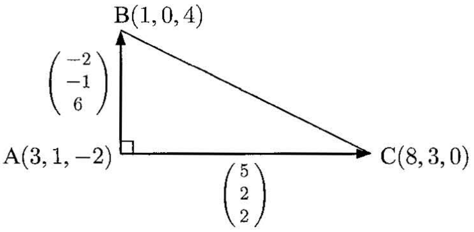
b $\overrightarrow{\text{BA}} = \begin{pmatrix}2\\1\\-6\end{pmatrix}$, $\overrightarrow{\text{BC}} = \begin{pmatrix}7\\3\\-4\end{pmatrix}$.

If we let θ be the angle ABC, then

$$\begin{pmatrix}2\\1\\-6\end{pmatrix} \bullet \begin{pmatrix}7\\3\\-4\end{pmatrix} = \sqrt{4+1+36}\sqrt{49+9+16}\cos\theta$$

$$\therefore\ 14+3+24=\sqrt{41}\sqrt{74}\cos\theta$$
$$\therefore\ \cos\theta=\frac{41}{\sqrt{41}\sqrt{74}}$$
$$\therefore\ \theta=\arccos\left(\frac{41}{\sqrt{41}\sqrt{74}}\right)\approx 41.9^\circ$$

or



From **a**, $\widehat{BAC}=90^\circ$

$$\text{so}\quad \tan\widehat{ABC}=\frac{\sqrt{5^2+2^2+2^2}}{\sqrt{(-2)^2+(-1)^2+6^2}}=\sqrt{\frac{33}{41}}$$

$\tan\widehat{ABC}\approx 0.8971$ and so $\widehat{ABC}\approx 41.9^\circ$

7 **a** In 2010, $u_1=4000$, $r=1.06$. In 2020, $n=11$.

$$\begin{aligned}u_{11}&=u_1r^{10}\\&=4000\times 1.06^{10}\\&\approx 7160\end{aligned}$$

About 7160 people are expected to visit the island.

b Total number of people

$$\begin{aligned}&=u_1+u_2+u_3+....+u_{10}+u_{11}\\&=S_{11}\\&=\frac{4000\times(1.06^{11}-1)}{1.06-1}\\&\approx 59\,886.57\end{aligned}$$

$\therefore$ the total charge $=59\,886.57\times\$5$
$\approx \$299\,000$

8 **a** f is defined providing $x>0$, $x+3>0$ and $x^2-9>0$.
$\therefore\ x>0,\ x>-3$ and $x<-3$ or $x>3$.
The three conditions are satisfied if $x>3$.
Domain of f is $x\in\mathbb{R},\ x>3$.

b
$$\begin{aligned}\ln x+\ln(x+3)-\ln(x^2-9)&=\ln\left(\frac{x(x+3)}{x^2-9}\right)\\&=\ln\left(\frac{x(x+3)}{(x-3)(x+3)}\right)\\&=\ln\left(\frac{x}{x-3}\right)\end{aligned}$$

c f is $y=\ln\left(\frac{x}{x-3}\right)$

$\therefore\ f^{-1}$ is given by $x=\ln\left(\frac{y}{y-3}\right)$

$$\begin{aligned}\therefore\ \frac{y}{y-3}&=e^x\\ \therefore\ y&=(y-3)e^x\\ \therefore\ y&=ye^x-3e^x\\ \therefore\ y(1-e^x)&=-3e^x\\ \therefore\ y&=\frac{3e^x}{e^x-1}\end{aligned}$$

So, $f^{-1}(x)=\dfrac{3e^x}{e^x-1}$.

9 Let $u=1-x$, $\dfrac{du}{dx}=-1$.

$$\begin{aligned}\therefore\quad &\int x^2\sqrt{1-x}\,dx\\&=\int(1-u)^2\sqrt{u}(-du)\\&=\int-(1-2u+u^2)u^{\frac{1}{2}}\,du\\&=\int(-u^{\frac{1}{2}}+2u^{\frac{3}{2}}-u^{\frac{5}{2}})\,du\\&=\frac{-u^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{7}{2}}}{\frac{7}{2}}+c\\&=-\tfrac{2}{3}(1-x)^{\frac{3}{2}}+\tfrac{4}{5}(1-x)^{\frac{5}{2}}-\tfrac{2}{7}(1-x)^{\frac{7}{2}}+c\end{aligned}$$

10 If $1+ai$ is a zero of x^2+ax+5, then so is $1-ai$.
These have sum 2 and product $1+a^2$ and so come from the quadratic equation $x^2-2x+(1+a^2)=0$
Comparing coefficients, $a=-2$ and $1+a^2=5$
So, the solution is $a=-2$.

11 Since -2 and 3 are x-intercepts, the polynomial has factors $(x+2)$ and $(x-3)$.
The polynomial touches the x-axis at 1, so $(x-1)^2$ is a factor.

So, $f(x)=a(x+2)(x-3)(x-1)^2$
But $f(0)=a(2)(-3)(1)=-12$ and so $a=2$

$$\begin{aligned}\text{Hence}\quad f(x)&=2(x+2)(x-3)(x-1)^2\\&=2(x^2-x-6)(x^2-2x+1)\\&=2x^4-6x^3-6x^2+22x-12\end{aligned}$$

12 A double is (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$\therefore$ P(double) $=\frac{6}{36}=\frac{1}{6}$.

P(no double in n throws) $=\binom{n}{0}(\frac{1}{6})^0(\frac{5}{6})^n=(\frac{5}{6})^n$.

$\therefore$ P(double in n throws) $=1-(\frac{5}{6})^n$

So, we need to find n such that $1-(\frac{5}{6})^n\geqslant 0.7$

$$\begin{aligned}\therefore\ (\tfrac{5}{6})^n&\leqslant 0.3\\ \therefore\ \log(\tfrac{5}{6})^n&\leqslant\log(0.3)\\ \therefore\ n\log(\tfrac{5}{6})&\leqslant\log(0.3)\\ \therefore\ n&\geqslant\frac{\log(0.3)}{\log(\frac{5}{6})}\\ \therefore\ n&\geqslant 6.6\end{aligned}$$

So, the pair of dice must be rolled at least 7 times.

13 Each of the nine tickets can be allocated in 2 ways, either to one student, or the other.
There are 2^9 ways of allocating 9 tickets if there are no restrictions.
There are 2 ways in which one student has no tickets allocated.
Total number of ways of allocating 9 tickets so that each student has a least 1 ticket is $2^9-2=510$ ways.

14
$$\begin{aligned}\mathrm{P}(|Z|\leqslant a)&=\mathrm{P}(-a\leqslant Z\leqslant a)\\&=1-2\,\mathrm{P}(Z\leqslant -a)\end{aligned}$$

$$\begin{aligned}\therefore\ 1-2\,\mathrm{P}(Z\leqslant -a)&=0.72\\ \therefore\ 2\,\mathrm{P}(Z\leqslant -a)&=0.28\\ \therefore\ \mathrm{P}(Z\leqslant -a)&=0.14\end{aligned}$$

But $Z\sim\mathrm{N}(0,\,1^2)$
$\therefore\ -a\approx -1.0803$
$\therefore\ a\approx 1.08$

15 $$x^2 + x\ln y - y = 3$$

$$\therefore\quad 2x + 1\ln y + x\left(\frac{1}{y}\right)\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\therefore\quad 2x + \ln y + \frac{x}{y}\frac{dy}{dx} - \frac{dy}{dx} = 0$$

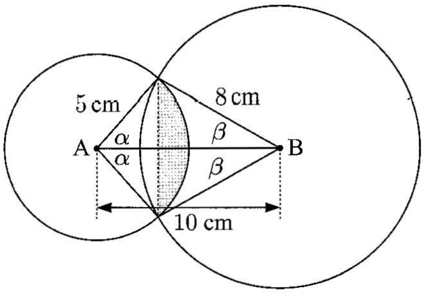
So, at (2, 1), $4 + 0 + 2\dfrac{dy}{dx} - \dfrac{dy}{dx} = 0$

$$\therefore\quad \frac{dy}{dx} = -4$$

So, the equation is $\quad y - 1 = -4(x - 2)$

$$\therefore\quad y = -4x + 9$$

16



Using the cosine rule: $\quad \cos\alpha = \dfrac{5^2 + 10^2 - 8^2}{2 \times 5 \times 10} = \frac{61}{100}$

$$\cos\beta = \frac{10^2 + 8^2 - 5^2}{2 \times 10 \times 8} = \tfrac{139}{160}$$

Thus, in radians, $\quad \alpha \approx 0.9147$ and $\beta \approx 0.5181$

$$\therefore\quad 2\alpha \approx 1.829 \text{ and } 2\beta \approx 1.036$$

Using area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$,

total area

$= \frac{1}{2} \times 5^2 \times (2\alpha - \sin 2\alpha) + \frac{1}{2} \times 8^2 \times (2\beta - \sin 2\beta)$

$= \frac{25}{2}(1.829 - \sin 1.829) + 32(1.036 - \sin 1.036)$

$\approx 16.4 \text{ cm}^2$

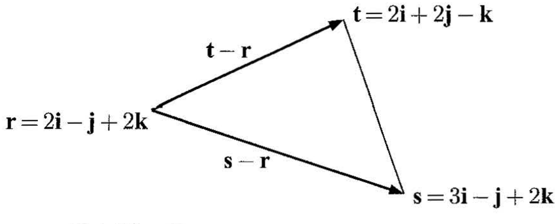
17 $v = e^t \cos 2t$

a Distance travelled in first 5 seconds $= \int_0^5 |v|\, dt$

$$= \int_0^5 \left|e^t \cos 2t\right| dt$$

b Using technology, this distance ≈ 108 m.

18



$\mathbf{t} - \mathbf{r} = 0\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$

$\mathbf{s} - \mathbf{r} = \mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

The area of the triangle is $\frac{1}{2}\left|(\mathbf{t} - \mathbf{r}) \times (\mathbf{s} - \mathbf{r})\right|$

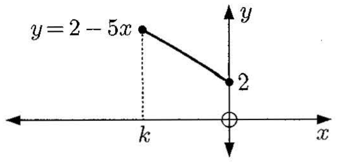
Now $\quad (\mathbf{t} - \mathbf{r}) \times (\mathbf{s} - \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -3 \\ 1 & 0 & 0 \end{vmatrix}$

$$= (0 - 0)\mathbf{i} - (0 + 3)\mathbf{j} + (0 - 3)\mathbf{k}$$
$$= 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

$\therefore\quad$ the area of the triangle $= \frac{1}{2}\sqrt{0^2 + (-3)^2 + (-3)^2}$

$$= \tfrac{1}{2}\sqrt{18} \text{ units}^2.$$

19 Graph of $f(x)$ is shown.



a Since f is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\therefore\quad \int_k^0 (2 - 5x)\, dx = 1$$

$$\therefore\quad \left[2x - \frac{5x^2}{2}\right]_k^0 = 1$$

$$\therefore\quad 0 - \left(2k - \frac{5k^2}{2}\right) = 1$$

$$\therefore\quad \tfrac{5}{2}k^2 - 2k = 1$$

$$\therefore\quad 5k^2 - 4k - 2 = 0$$

$$\therefore\quad k = \frac{4 \pm \sqrt{16 - 4(5)(-2)}}{10} = \frac{4 \pm \sqrt{56}}{10}$$

But $k < 0$ so $\quad k = \dfrac{4 - \sqrt{56}}{10} \approx -0.348\,33 \approx -0.348$

b The mean $\mu = \int_k^0 x(2 - 5x)\, dx$

$= \int_k^0 (2x - 5x^2)\, dx$

$= \left[\dfrac{2x^2}{2} - \dfrac{5x^3}{3}\right]_k^0$

$= \frac{5}{3}k^3 - k^2$

$\approx \frac{5}{3}(-0.348\,33)^3 - (-0.348\,33)^2$

≈ -0.192

20 **a** Let X be the number of mistakes the typist makes in 1500 words. Since the typist makes an average of $m = 6$ errors per 1500 words we assume that $X \sim \text{Po}(6)$.

Hence, $\text{P}(X \leqslant 5) \approx 0.446$

b $\text{P}(X \leqslant 5 \mid X \geqslant 1) = \dfrac{\text{P}(X \leqslant 5 \cap X \geqslant 1)}{\text{P}(X \geqslant 1)}$

$$= \frac{\text{P}(1 \leqslant X \leqslant 5)}{\text{P}(X \geqslant 1)}$$

$$= \frac{\text{P}(X \leqslant 5) - \text{P}(X = 0)}{1 - \text{P}(X = 0)}$$

$$= \frac{0.445\,680 - 0.002\,479}{1 - 0.002\,479}$$

$$\approx 0.444$$

21 **a** $$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 9 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$$

and $\overrightarrow{AB} \bullet \overrightarrow{AC} = -10 + (-2) + 12 = 0$

So, $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.

b A vector $\mathbf{n}$ normal to the plane is given by $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 6 \\ 5 & 2 & 2 \end{vmatrix}$$

$$= (-2 - 12)\mathbf{i} - (-4 - 30)\mathbf{j} + (-4 + 5)\mathbf{k}$$
$$= -14\mathbf{i} + 34\mathbf{j} + \mathbf{k}$$

The equation of the plane is:

$$\begin{pmatrix}-14\\34\\1\end{pmatrix}\bullet\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-14\\34\\1\end{pmatrix}\bullet\begin{pmatrix}4\\2\\-1\end{pmatrix}$$

$\therefore\ -14x+34y+z=-56+68-1$

$\therefore\ -14x+34y+z=11$

The distance from $(8, 1, 0)$ to the plane is

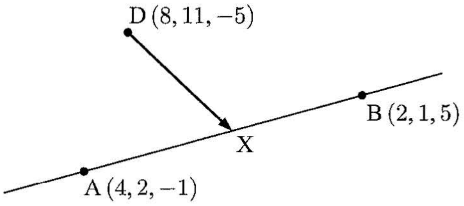
$$\frac{|-14\times8+34\times1+1\times0-11|}{\sqrt{14^2+34^2+1^2}}\approx 2.42\text{ units.}$$

c $\overrightarrow{AB}=\begin{pmatrix}-2\\-1\\6\end{pmatrix}$

An equation of a line through A and B is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-2\\-1\\6\end{pmatrix}.$$

d



Let X be a point on the line through A and B.

From **c**, $\overrightarrow{DX}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-2\\-1\\6\end{pmatrix}-\begin{pmatrix}8\\11\\-5\end{pmatrix}$

$=\begin{pmatrix}-4-2\lambda\\-9-\lambda\\4+6\lambda\end{pmatrix}$

$\therefore\ \overrightarrow{DX}\bullet\overrightarrow{AB}=\begin{pmatrix}-4-2\lambda\\-9-\lambda\\4+6\lambda\end{pmatrix}\bullet\begin{pmatrix}-2\\-1\\6\end{pmatrix}$

$=8+4\lambda+9+\lambda+24+36\lambda$

$=41+41\lambda$

$\overrightarrow{DX}$ is perpendicular to $\overrightarrow{AB}$ if $41+41\lambda=0\quad\therefore\ \lambda=-1$

So, $\overrightarrow{DX}=\begin{pmatrix}-2\\-8\\-2\end{pmatrix}$ and $|\overrightarrow{DX}|=\sqrt{4+64+4}$

$=\sqrt{72}$

The distance from D to the line is $\sqrt{72}$ units.

22 $f(x)=e^{\sin^2x},\quad 0\leqslant x\leqslant\pi$

a $f'(x)=e^{\sin^2x}\times2\sin x\cos x=\sin2x\,e^{\sin^2x}$

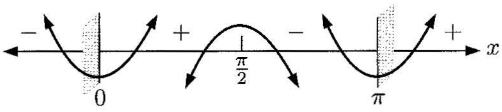
b When $f'(x)=0,\ \sin2x=0$

$\therefore\ 2x=0+k\pi$

$\therefore\ x=k\frac{\pi}{2}$

$\therefore\ x=0,\ \frac{\pi}{2},\ \pi$

Sign diagram of $f'(x)$:



$\therefore$ local minimum at $(0, 1)$ and $(\pi, 1)$

local maximum at $(\frac{\pi}{2}, e)$.

c $f''(x)=2\cos2xe^{\sin^2x}+\sin2x\sin2xe^{\sin^2x}$

$=e^{\sin^2x}[2\cos2x+\sin^22x]$

d At $x=\frac{3\pi}{4},\ \sin x=\frac{1}{\sqrt2},\ \sin2x=-1$

$\therefore\ f'(x)=-1e^{\frac12}=-\sqrt{e}$

$\therefore$ the tangent at $\left(\frac{3\pi}{4},\sqrt{e}\right)$ has equation

$y-\sqrt{e}=-\sqrt{e}\left(x-\frac{3\pi}{4}\right)$

$\therefore\ y=-\sqrt{e}x+\sqrt{e}+\dfrac{3\pi\sqrt{e}}{4}$

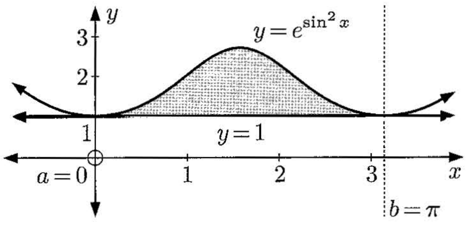
e The tangent cuts the x-axis when $y=0$

$\therefore\ 0=-\sqrt{e}x+\sqrt{e}+\dfrac{3\pi\sqrt{e}}{4}$

$\therefore\ 0=-x+1+\frac{3\pi}{4}$

$\therefore\ x=1+\frac{3\pi}{4}$

f



Using technology,

area $=\int_0^\pi\left(e^{\sin^2x}-1\right)dx\approx2.37$ units^2

23 $X\sim N(90,\sigma^2)$.

Now $P(X<88)=0.289\,25$

$\therefore\ P\left(\dfrac{X-90}{\sigma}<\dfrac{88-90}{\sigma}\right)=0.28925$

$\therefore\ P\left(Z<\dfrac{-2}{\sigma}\right)=0.289\,25$

$\therefore\ -\dfrac{2}{\sigma}\approx-0.5556$

$\therefore\ \sigma\approx3.60$

So, $X\sim N(90, 3.60^2)$.

Hence $P(90<X<92)\approx0.210$

About 21.0% of the scores lie between 90 and 92.

24 **a** **i** P_n is: "$\sum\limits_{r=1}^{n}r^3=\dfrac{n^2(n+1)^2}{4}$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, LHS $=1^3=1$,

RHS $=\dfrac{1^2\times2^2}{4}=1$

$\therefore\ P_1$ is true.

(2) If P_k is true,

$1^3+2^3+3^3+....+k^3=\dfrac{k^2(k+1)^2}{4}$

$\therefore\ 1^3+2^3+3^3+....+k^3+(k+1)^3$

$=\dfrac{k^2(k+1)^2}{4}+(k+1)^3$

$=\dfrac{k^2(k+1)^2+4(k+1)^3}{4}$

$=\dfrac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$

$=\dfrac{(k+1)^2(k+2)^2}{4}$

$=\dfrac{(k+1)^2([k+1]+1)^2}{4}$

Thus P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore\ P_n$ is true for $n\in\mathbb{Z}^+$

{Principle of mathematical induction}

ii $1^3+2^3+....+100^3=\dfrac{100^2(101)^2}{4}=25\,502\,500$

b $\sum\limits_{r=1}^{n} r=1+2+3+....+n$

which is arithmetic with $u_1=1,\ d=1.$

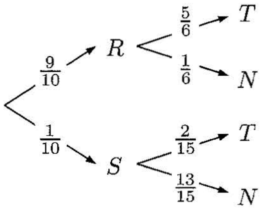
$$\therefore\ \sum_{r=1}^{n} r=\frac{n}{2}\left[2u_1+(n-1)d\right]$$
$$=\frac{n}{2}\left[2+n-1\right]$$
$$=\frac{n(n+1)}{2}$$

So, $\left(\sum\limits_{r=1}^{n} r\right)^2=\dfrac{n^2(n+1)^2}{4}=\sum\limits_{r=1}^{n} r^3$ {from **a i**}

25 The following tree diagram shows what is happening.

R: alarm rings $\quad$ S: alarm does not ring

T: arrives for training $\quad$ N: does not arrive for training



a $\mathrm{P}(T)=\frac{9}{10}\times\frac{5}{6}+\frac{1}{10}\times\frac{2}{15}=\frac{229}{300}$

b $\mathrm{P}(S\mid N)=\dfrac{\mathrm{P}(S\cap N)}{\mathrm{P}(N)}=\dfrac{\frac{1}{10}\times\frac{13}{15}}{\frac{9}{10}\times\frac{1}{6}+\frac{1}{10}\times\frac{13}{15}}=\frac{26}{71}$

SOLUTIONS TO EXAMINATION PRACTICE SET 9

1 **a**
$$(a+3i)(b-i)=13+i$$
$$\therefore\ [ab+3]+[3b-a]i=13+i$$
$$\therefore\ ab+3=13 \text{ and } 3b-a=1$$
$$\therefore\ ab=10 \text{ and } 3b-a=1$$
$$\therefore\ 3b-\frac{10}{b}=1$$
$$\therefore\ 3b^2-10=b$$
$$\therefore\ 3b^2-b-10=0$$
$$\therefore\ (3b+5)(b-2)=0$$
$$\therefore\ b=-\tfrac{5}{3} \text{ or } 2$$

But $b\in\mathbb{Z}$, so $b=2$

$$\therefore\ a=5$$

b
$$|z+1+i|=2\,|z-2-i|$$
$$\therefore\ |5+ai+1+i|=2\,|5+ai-2-i|$$
$$\therefore\ |6+(a+1)i|=2\,|3+(a-1)i|$$
$$\therefore\ \sqrt{36+(a+1)^2}=2\sqrt{9+(a-1)^2}$$
$$\therefore\ 36+(a+1)^2=4\left[9+(a-1)^2\right]$$
$$\therefore\ 36+a^2+2a+1=36+4a^2-8a+4$$
$$\therefore\ 3a^2-10a+3=0$$
$$\therefore\ (3a-1)(a-3)=0 \text{ and so } a=\tfrac{1}{3} \text{ or } 3$$

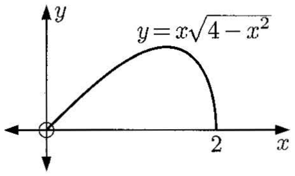
But $a\in\mathbb{Z}$, so $a=3$

2 **a** Let the arithmetic sequence have first term u_1 and common difference d.

sum of even terms
$$=u_2+u_4+u_6+u_8+....+u_{30}$$
$$=u_1+d+u_3+d+u_5+d+....+u_{29}+d$$
$$=\text{sum of odd terms}+15d$$
$$\therefore\ 15d=8 \text{ and so } d=\tfrac{8}{15}$$

b
$$S_n=\frac{n}{2}\left[2u_1+(n-1)d\right]$$
$$=\frac{n}{2}\left[2u_1+\frac{8(n-1)}{15}\right]$$
$$=n\left(u_1+\frac{4(n-1)}{15}\right)$$

3 Area $=\int_0^2 x\sqrt{4-x^2}\,dx$



Let $u=4-x^2,\ \dfrac{du}{dx}=-2x.$

When $x=0,\ u=4.$

When $x=2,\ u=0.$

$$\therefore\ \text{area}=\int_0^2\sqrt{u}\left(-\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$=\int_4^0-\tfrac{1}{2}u^{\frac{1}{2}}\,du$$
$$=\tfrac{1}{2}\int_0^4 u^{\frac{1}{2}}\,du$$
$$=\tfrac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4=\tfrac{1}{3}\left[u^{\frac{3}{2}}\right]_0^4$$
$$=\tfrac{1}{3}\left(4^{\frac{3}{2}}-0\right)$$
$$=\tfrac{8}{3}\text{ units}^2$$

4 $f:x\mapsto\sqrt{x},\quad g:x\mapsto 1-\sin x$

Domain of f is all $x\geqslant 0$.

Domain of $f\circ g=f(g(x))$ is all x such that $g(x)\geqslant 0$.

$\therefore\ 1-\sin x\geqslant 0$

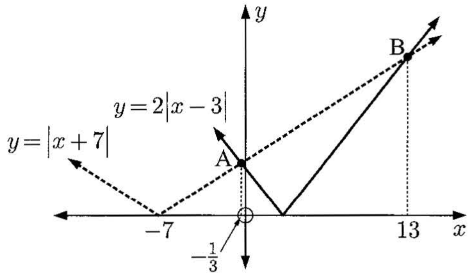
$\therefore\ \sin x\leqslant 1$ which is true for all $x\in\mathbb{R}$.

$\therefore$ the domain of $f\circ g$ is $\{x\mid x\in\mathbb{R}\}$.

As $f\circ g=\sqrt{1-\sin x}$, range of $f\circ g$ is $\{y\mid 0\leqslant y\leqslant\sqrt{2}\}$.

5 The graphs meet where $2\,|x-3|=|x+7|$
$$\therefore\ 4(x-3)^2-(x+7)^2=0$$
$$\therefore\ (2x-6+x+7)\,(2x-6-x-7)=0$$
$$\therefore\ (3x+1)\,(x-13)=0$$
$$\therefore\ x=-\tfrac{1}{3} \text{ or } 13$$

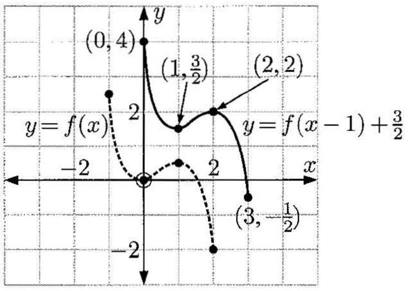
We sketch $y=2\,|x-3|$ and $y=|x+7|$.



$2\,|x-3|\leqslant|x+7|$ when the graph of $y=2\,|x-3|$ is below the graph of $y=|x+7|$.

This is between A and B. So, $-\frac{1}{3}\leqslant x\leqslant 13$.

6 **a**



b Local minimum of $y = f(x-1) + \frac{3}{2}$ is $\left(1, \frac{3}{2}\right)$.

Local maximum of $y = f(x-1) + \frac{3}{2}$ is $(2, 2)$.

7
$$\begin{aligned}&\sin\left(\tfrac{\pi}{2} - \theta\right) \cos\left(\tfrac{\pi}{2} + \theta\right) \csc(\pi - 2\theta)\\ &= \cos\theta \times (-\sin\theta) \times \frac{1}{\sin(\pi - 2\theta)}\\ &= -\cos\theta \sin\theta \times \frac{1}{\sin 2\theta}\\ &= -\sin\theta\cos\theta \times \frac{1}{2\sin\theta\cos\theta}\\ &= -\tfrac{1}{2}\end{aligned}$$

8
$$\begin{aligned}&\int \sin^3 x \cos^3 x \, dx\\ &= \int \sin^3 x\,(1 - \sin^2 x) \cos x \, dx\\ &= \int (\sin^3 x - \sin^5 x) \cos x \, dx\\ &= \int (u^3 - u^5)\, du \qquad \left\{\text{letting } u = \sin x, \ \frac{du}{dx} = \cos x\right\}\\ &= \frac{u^4}{4} - \frac{u^6}{6} + c\\ &= \tfrac{1}{4}\sin^4 x - \tfrac{1}{6}\sin^6 x + c\end{aligned}$$

9 **a** When $x = -2$,
$$\begin{aligned}3(-2)^2 + 2(-2)y - y^2 &= 7\\ \therefore\ 12 - 4y - y^2 &= 7\\ \therefore\ y^2 + 4y - 5 &= 0\\ \therefore\ (y-1)(y+5) &= 0\\ \therefore\ y &= 1 \text{ or } -5\end{aligned}$$

$\therefore$ the points are $(-2, 1)$ and $(-2, -5)$.

b Now
$$\begin{aligned}6x + \left[2y + 2x\frac{dy}{dx}\right] - 2y\frac{dy}{dx} &= 0\\ \therefore\ 6x + 2y + (2x - 2y)\frac{dy}{dx} &= 0\end{aligned}$$

At $(-2, 1)$,
$$\begin{aligned}-12 + 2 + (-6)\frac{dy}{dx} &= 0\\ \therefore\ 6\frac{dy}{dx} &= -10\\ \therefore\ \frac{dy}{dx} &= -\tfrac{5}{3}\end{aligned}$$

$\therefore$ the tangent has gradient $-\frac{5}{3}$ and the normal has gradient $\frac{3}{5}$

$\therefore$ the equation of the normal is $y - 1 = \frac{3}{5}(x+2)$, which is $5y - 5 = 3x + 6$

or $3x - 5y = -11$.

or at $(-2, -5)$, $\dfrac{dy}{dx} = \frac{22}{6} = \frac{11}{3}$

$\therefore$ the normal has gradient $-\frac{3}{11}$

$\therefore$ the normal is $y + 5 = -\frac{3}{11}(x+2)$

which is $3x + 11y = -61$

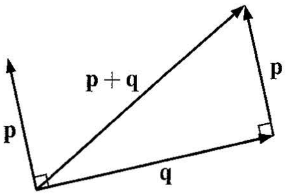
10 **a** $|\mathbf{p} + \mathbf{q}| = \sqrt{(\mathbf{p}+\mathbf{q}) \bullet (\mathbf{p}+\mathbf{q})} = \sqrt{25} = 5$

b
$$\begin{aligned}(\mathbf{p}+\mathbf{q}) \bullet (\mathbf{p}+\mathbf{q}) &= \mathbf{p}\bullet\mathbf{p} + \mathbf{p}\bullet\mathbf{q} + \mathbf{q}\bullet\mathbf{p} + \mathbf{q}\bullet\mathbf{q}\\ &= |\mathbf{p}|^2 + 2(\mathbf{p}\bullet\mathbf{q}) + |\mathbf{q}|^2\end{aligned}$$

$\therefore\ 25 = 25 + 2(\mathbf{p} \bullet \mathbf{q})$ {given}

and $\mathbf{p} \bullet \mathbf{q} = 0$

c

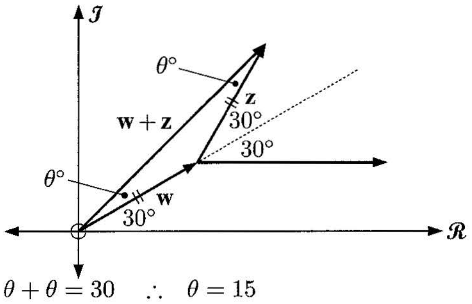


Note that since $\mathbf{p} \bullet \mathbf{q} = 0$, $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

11 $|w| = |z|$, $\arg w = \frac{\pi}{6}$, $\arg z = \frac{\pi}{3}$

a $\arg(wz) = \arg w + \arg z = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

b $w + z = R\,\text{cis}\left(\frac{\pi}{6}\right) + R\,\text{cis}\left(\frac{\pi}{3}\right)$



$\theta + \theta = 30 \quad \therefore\ \theta = 15$

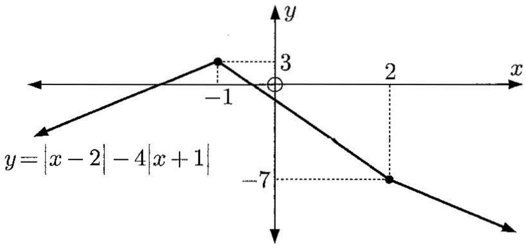
$\therefore\ \arg(w+z) = (30+\theta)^\circ = 45^\circ = \frac{\pi}{4}$

12 f is defined for all x $\therefore$ domain is all $x \in \mathbb{R}$

If $x \geqslant 2$,
$$\begin{aligned}|x-2| - 4|x+1| &= (x-2) - 4(x+1)\\ &= -3x - 6\end{aligned}$$

If $-1 \leqslant x \leqslant 2$,
$$\begin{aligned}|x-2| - 4|x+1| &= -(x-2) - 4(x+1)\\ &= -5x - 2\end{aligned}$$

If $x \leqslant -1$,
$$\begin{aligned}|x-2| - 4|x+1| &= -(x-2) - 4(-[x+1])\\ &= 3x + 6\end{aligned}$$



f has a maximum value of 3 which occurs when $x = -1$.
So, range of f is $y \leqslant 3$.

13 Since $xy = 2$, $y = \dfrac{2}{x}$ provided $x \neq 0$.

The graphs meet where
$$\begin{aligned}x^3 + \left(\frac{2}{x}\right)^3 &= 9\\ \therefore\ x^3 + \frac{8}{x^3} &= 9\\ \therefore\ x^6 + 8 &= 9x^3\\ \therefore\ x^6 - 9x^3 + 8 &= 0\end{aligned}$$

Letting $x^3 = z, \quad z^2 - 9z + 8 = 0$

$\therefore \quad (z-8)(z-1) = 0$

$\therefore \quad z = 8 \text{ or } 1$

$\therefore \quad x^3 = 8 \text{ or } 1$

$\therefore \quad x = 2 \text{ or } 1$

So, $x = 2, \ y = 1$ or $x = 1, \ y = 2$.

14 The equation $6x^2 + px + q = 0$ has roots 2 and $-\frac{1}{2}$.

The sum of the roots $= \frac{3}{2}$

$\therefore \quad -\dfrac{p}{6} = \frac{3}{2}$ and so $p = -9$.

The product of the roots is -1

$\therefore \quad \dfrac{q}{6} = -1$ and so $q = -6$.

15 $\tan\beta = \frac{2}{3}, \quad \pi < \beta < \frac{3\pi}{2}$

Now $\quad \tan\beta = \dfrac{2\tan\left(\frac{\beta}{2}\right)}{1 - \tan^2\left(\frac{\beta}{2}\right)}$

$\therefore \quad \dfrac{2t}{1-t^2} = \frac{2}{3} \qquad \{\text{letting } t = \tan(\tfrac{\beta}{2})\}$

$\therefore \quad 3t = 1 - t^2$

$\therefore \quad t^2 + 3t - 1 = 0$

$\therefore \quad t = \dfrac{-3 \pm \sqrt{9 - 4(1)(-1)}}{2}$

$\therefore \quad t = \dfrac{-3 \pm \sqrt{13}}{2}$

But $\frac{\pi}{2} < \frac{\beta}{2} < \frac{3\pi}{4}$

$\therefore \quad \frac{\beta}{2}$ lies in quadrant 2 and so $\tan\left(\frac{\beta}{2}\right)$ is negative.

$\therefore \quad \tan\left(\frac{\beta}{2}\right) = \dfrac{-3 - \sqrt{13}}{2}$

16 Graph is $\quad y = a\sin bx + c$

$y(0) = c = 1$

period $= \dfrac{2\pi}{b} = \pi \quad$ so $\quad b = 2$

amplitude $= |a| = 3 \quad$ so $\quad a = 3$.

a $a = 3$ **b** $b = 2$ **c** $c = 1$

17 $y = x^2 \ln\left(\dfrac{1}{x^2}\right) = x^2 \ln x^{-2}$

$\therefore \quad y = -2x^2 \ln x \qquad \{x > 0\}$

So, $\quad \dfrac{dy}{dx} = -4x\ln x - 2x^2\left(\dfrac{1}{x}\right)$

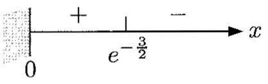
$= -4x\ln x - 2x$

and $\quad \dfrac{d^2y}{dx^2} = -4\ln x - 4x\left(\dfrac{1}{x}\right) - 2$

$= -4\ln x - 6$

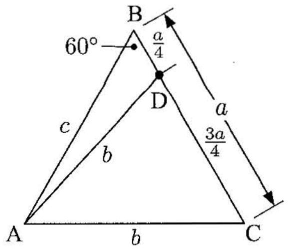
So, $\dfrac{d^2y}{dx^2} = 0$ when $\ln x = -\frac{3}{2} \quad \therefore \quad x = e^{-\frac{3}{2}}$

The sign diagram for $\dfrac{d^2y}{dx^2}$ is:



$\therefore$ the point of inflection has x-coordinate $e^{-\frac{3}{2}}$.

18 **a**



As $\quad$ BD : DC $= 1 : 3$, $\quad$ BD $= \dfrac{a}{4}$, $\quad$ DC $= \dfrac{3a}{4}$

Using the cosine rule:

In $\triangle$ABC, $\quad b^2 = c^2 + a^2 - 2ac\cos 60°$

$\therefore \quad b^2 = c^2 + a^2 - ac \quad .... (1)$

In $\triangle$ABD, $\quad b^2 = c^2 + \left(\dfrac{a}{4}\right)^2 - 2c\left(\dfrac{a}{4}\right)\cos 60°$

$\therefore \quad b^2 = c^2 + \dfrac{a^2}{16} - \dfrac{ac}{4} \quad .... (2)$

b Using (1) and (2),

$c^2 + a^2 - ac = c^2 + \dfrac{a^2}{16} - \dfrac{ac}{4}$

$\therefore \quad 16a^2 - 16ac = a^2 - 4ac$

$\therefore \quad 15a^2 - 12ac = 0$

$\therefore \quad 3a(5a - 4c) = 0$

$\therefore \quad 5a = 4c$

$\therefore \quad a = \frac{4}{5}c$

19 $\quad f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 + 5(x+h) - 2 - (3x^2 + 5x - 2)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 5x + 5h - 2 - 3x^2 - 5x + 2}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 5h}{h}$$

$$= \lim_{h\to 0} 6x + 5 + 3h$$

$$= 6x + 5$$

20 We integrate by parts with $\quad u = e^x \qquad v' = \sin x$

$u' = e^x \qquad v = -\cos x$

$\therefore \quad \int e^x \sin x\, dx = -e^x\cos x - \int -e^x \cos x\, dx$

$= -e^x\cos x + \int e^x\cos x\, dx$

We again integrate by parts, now with $\quad u = e^x \qquad v' = \cos x$

$u' = e^x \qquad v = \sin x$

$\therefore \quad \int e^x \sin x\, dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\, dx + c'$

$\therefore \quad 2\int e^x\sin x\, dx = e^x(\sin x - \cos x) + c'$

$\therefore \quad \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + c$

21 **a** If **a** and **b** are perpendicular then $\mathbf{a} \bullet \mathbf{b} = 0$

$\therefore \quad -t + 2(1+t) - 2(2t) = 0$

$\therefore \quad -t + 2 + 2t - 4t = 0$

$\therefore \quad -3t + 2 = 0$ and so $t = \frac{2}{3}$

b The vectors are parallel if $\quad \dfrac{-t}{1} = \dfrac{1+t}{2} = \dfrac{2t}{-2}$

$\therefore \quad -2t = 1 + t$

$\therefore \quad t = -\frac{1}{3}$

22 $$2\sin^2 x - \sin x - 2\sin x\cos x + \cos x = 0$$

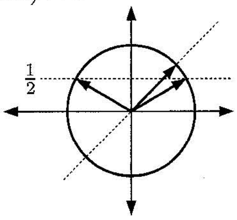
$$\therefore\ \sin x(2\sin x - 1) - \cos x(2\sin x - 1) = 0$$

$$\therefore\ (2\sin x - 1)(\sin x - \cos x) = 0$$

$$\therefore\ \sin x = \tfrac{1}{2} \quad \text{or} \quad \sin x = \cos x$$

$$\therefore\ \sin x = \tfrac{1}{2} \quad \text{or} \quad \tan x = 1$$

$$\therefore\ x = \tfrac{\pi}{6}, \tfrac{\pi}{4}, \tfrac{5\pi}{6}$$

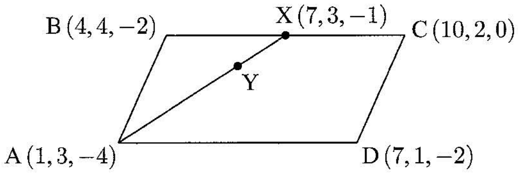


23 **a** $\overrightarrow{BC} = \begin{pmatrix} 10-4 \\ 2-4 \\ 0-(-2) \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}$

Since $\overrightarrow{BC} = \overrightarrow{AD}$, position vector of D is

$\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -2 \end{pmatrix}$ and the coordinates of D are (7, 1, −2).

The midpoint of [BC] is $\left(\frac{4+10}{2}, \frac{4+2}{2}, \frac{-2+0}{2}\right)$, so the coordinates of X are (7, 3, −1).



The ratio AY : YX = 2 : 1, so position vector of Y is

$$\tfrac{1}{3}\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

The coordinates of Y are (5, 3, −2).

b $\overrightarrow{BY} = \begin{pmatrix} 5-4 \\ 3-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} 7-4 \\ 1-4 \\ -2--2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$

Since $\overrightarrow{BY} = \frac{1}{3}\overrightarrow{BD}$, $\overrightarrow{BY}$ and $\overrightarrow{BD}$ are parallel

∴ B, Y, and D are collinear.

24 **a** **i** $1 - i\sqrt{3}$ has $\left|1 - i\sqrt{3}\right| = \sqrt{1+3} = 2$

$$\therefore\ 1 - i\sqrt{3} = 2\left(\tfrac{1}{2} - \tfrac{i\sqrt{3}}{2}\right)$$

$$= 2\,\text{cis}\left(-\tfrac{\pi}{3}\right)$$

$1 - i$ has $|1 - i| = \sqrt{1+1} = \sqrt{2}$

$$\therefore\ 1 - i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)$$

$$= \sqrt{2}\,\text{cis}\left(-\tfrac{\pi}{4}\right)$$

ii $$\frac{(1-i\sqrt{3})^{11}}{(1-i)^{18}} = \frac{\left[2\ \text{cis}\left(-\frac{\pi}{3}\right)\right]^{11}}{\left[\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)\right]^{18}}$$

$$= \frac{2^{11}\ \text{cis}\left(-\frac{11\pi}{3}\right)}{2^{9}\ \text{cis}\left(-\frac{9\pi}{2}\right)}$$

$$= 2^2\ \text{cis}\left(-\tfrac{11\pi}{3} - -\tfrac{9\pi}{2}\right)$$

$$= 4\,\text{cis}\left(\tfrac{9\pi}{2} - \tfrac{11\pi}{3}\right)$$

$$= 4\,\text{cis}\left(\tfrac{5\pi}{6}\right)$$

$$= 4\cos\left(\tfrac{5\pi}{6}\right) + 4\sin\left(\tfrac{5\pi}{6}\right)i$$

$$= 4\left(-\tfrac{\sqrt{3}}{2}\right) + 4\left(\tfrac{1}{2}\right)i$$

$$= -2\sqrt{3} + 2i$$

b $z^5 = \sqrt{3} - i$ has $|\sqrt{3} - i| = \sqrt{3+1} = 2$

$$\therefore\ z^5 = 2\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)$$

$$= 2\,\text{cis}\left(-\tfrac{\pi}{6}\right)$$

$$= 2\,\text{cis}\left(-\tfrac{\pi}{6} + k2\pi\right), \quad k \in \mathbb{Z}$$

$$\therefore\ z = 2^{\frac{1}{5}}\text{cis}\left(\frac{-\frac{\pi}{6} + k2\pi}{5}\right), \quad k \in \mathbb{Z} \quad \{\text{De Moivre}\}$$

$$\therefore\ z = \sqrt[5]{2}\,\text{cis}\left(-\tfrac{\pi}{30}\right),\ \sqrt[5]{2}\,\text{cis}\left(\tfrac{11\pi}{30}\right),\ \sqrt[5]{2}\,\text{cis}\left(\tfrac{23\pi}{30}\right),$$
$$\sqrt[5]{2}\,\text{cis}\left(-\tfrac{25\pi}{30}\right),\ \sqrt[5]{2}\,\text{cis}\left(-\tfrac{13\pi}{30}\right).$$

25 **a** $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\therefore\ f'(x)$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x} + e^x - e^{-x})(e^x + e^{-x} - e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(2e^x)(2e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} \quad \text{which is never zero.}$$

So, no stationary points exist.

b $$f'(x) = 4(e^x + e^{-x})^{-2}$$

$$\therefore\ f''(x) = -8(e^x + e^{-x})^{-3}(e^x - e^{-x})$$

$$= \frac{-8(e^x - e^{-x})}{(e^x + e^{-x})^3}$$

c The integral $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ has the form

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ where $f(x) = e^x + e^{-x}$.

$$\therefore\ \int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \left[\ln\left|e^x + e^{-x}\right|\right]_0^{\ln 3}$$

$$= \ln\left(3 + \tfrac{1}{3}\right) - \ln(1+1)$$

$$= \ln\left(\tfrac{10}{3}\right) - \ln 2$$

$$= \ln\left(\tfrac{5}{3}\right)$$

SOLUTIONS TO EXAMINATION PRACTICE SET 10

1 Growth is geometric with $u_1 = 1250$ and $r = 1.03$

We need to find n such that $u_n > 2000$.

$$\therefore\ u_1 r^{n-1} > 2000$$

$$\therefore\ 1250(1.03)^{n-1} > 2000$$

$$\therefore\ (1.03)^{n-1} > 1.6$$

$$\therefore\ \log(1.03)^{n-1} > \log(1.6)$$

$$\therefore\ n - 1 > \frac{\log(1.6)}{\log(1.03)}$$

$$\therefore\ n > \frac{\log(1.6)}{\log(1.03)} + 1$$

$\therefore\ n > 16.9$

$\therefore\ n = 17, 18, 19, \ldots$

u_1 corresponds to year 2005.

u_{17} corresponds to year 2021.

So, the population will first exceed 2000 in the year 2021.

2 Normal has vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and line has vector $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

If θ is the acute angle between the normal and the line, then

$$\cos\theta = \frac{\left|\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\right|}{\sqrt{4+4+1}\sqrt{1+4+1}} = \frac{1}{3\sqrt{6}}$$

$\therefore\ \theta \approx 82.18°$

The angle between the plane and the line is $90° - 82.18° \approx 7.82°$.

3 **a** There are 4 odd digits.

There are 4 odd digits to select from at the start, leaving 3 odd digits to end.

This leaves 5 digits from which to select the remaining 3.

As order matters, this can be done in $5 \times 4 \times 3$ ways.

Total number is $4 \times 3 \times (5 \times 4 \times 3) = 720$ ways.

b Starting with an even digit, the number of arrangements can be represented in the diagram:

even	odd	even	odd	even
3	4	2	3	1

Total is 72.

Starting with an odd digit we have

odd	even	odd	even	odd
4	3	3	2	2

Total number of ways is 144.

Total number of ways of alternating is $72 + 144 = 216$.

4 $\quad \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

But $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$ $\{A$ and B independent$\}$

So, $0.8 = 0.27 + \mathrm{P}(B) - 0.27\,\mathrm{P}(B)$

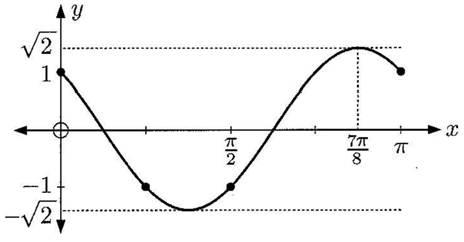
$\therefore\ 0.53 = (1 - 0.27)\,\mathrm{P}(B)$

$\therefore\ \mathrm{P}(B) = \frac{0.53}{0.73}$

≈ 0.726

5 $f(x) = \cos 2x - \sin 2x,\ 0 \leqslant x \leqslant \pi$

a $f(0) = 1,\quad f(\pi) = 1$



b Using technology, the maximum value is $\sqrt{2}$ when $x = \frac{7\pi}{8}$.

6 $\quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$

Hence $\sin\theta = \frac{|12\mathbf{j} - 5\mathbf{k}|}{4 \times 5} = \frac{\sqrt{12^2 + 5^2}}{20} = \frac{13}{20}$

and $\theta \approx 40.54°$ or $139.46°$

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

$\approx 4 \times 5 \times \cos(40.54°) \approx 15.2$

or $\mathbf{a} \bullet \mathbf{b} \approx 4 \times 5 \times \cos(139.46°) \approx -15.2$

7 **a** Using the midpoint of each time interval, the mean time

$$\bar{t} = \frac{\sum ft}{\sum f} \approx 50.25$$

b The variance $s_n^2 = \frac{\sum f(t - \bar{t})^2}{\sum f} \approx 519.94$

Hence, $s_n = \sqrt{519.94} \approx 22.80$

8 **a** Total number of ways of selecting 5 from 12 is $\binom{12}{5}$.

If no men are selected, there are $\binom{7}{5}$ ways of selecting 5 women.

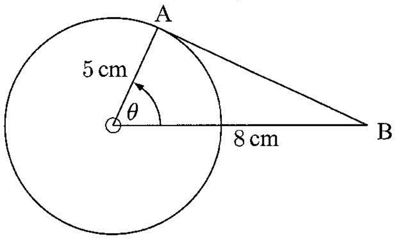
The probability of selecting 5 women is $\frac{\binom{7}{5}}{\binom{12}{5}} \approx 0.0265$.

b Number of ways of selecting 3 women and 2 men is $\binom{7}{3}\binom{5}{2}$.

So, the probability of selecting 3 women and 2 men is

$$\frac{\binom{7}{3}\binom{5}{2}}{\binom{12}{5}} \approx 0.442.$$

9



$$\frac{d\theta}{dt} = \frac{1\text{ rev}}{2\text{ s}} = \frac{2\pi^c}{2\text{ s}} = \pi^c\ \text{s}^{-1}$$

Area of $\triangle OAB = \frac{1}{2}(5)(8)\sin\theta$

$\therefore\ A = 20\sin\theta$

$\therefore\ \frac{dA}{dt} = 20\cos\theta\,\frac{d\theta}{dt}$

Particular case: $\theta = \frac{\pi}{6}$

$\therefore\ \frac{dA}{dt} = 20 \times \frac{\sqrt{3}}{2}(\pi)$

$= 10\sqrt{3}\pi\ \text{cm}^2\,\text{s}^{-1}$

So, the area is increasing at $10\pi\sqrt{3}\ \text{cm}^2\,\text{s}^{-1}$.

10 **a** $\int \tan^5 x\, dx$

$= \int (\tan^3 x)(\tan^2 x)\, dx$

$= \int \tan^3 x(\sec^2 x - 1)\, dx$

$= \int [\tan x]^3 \sec^2 x\, dx - \int \tan^3 x\, dx$

Letting $u = \tan x,\ \frac{du}{dx} = \sec^2 x,$

$\int \tan^5 x\, dx$

$= \int u^3 \frac{du}{dx}\, dx - \int \tan x \tan^2 x\, dx$

$= \frac{u^4}{4} - \int \tan x(\sec^2 x - 1)\, dx$

$= \frac{1}{4}\tan^4 x - \int \tan x \sec^2 x\, dx + \int \tan x\, dx$

$= \frac{1}{4}\tan^4 x - \int u \frac{du}{dx}\, dx - \int \frac{-\sin x}{\cos x}\, dx$

$= \frac{1}{4}\tan^4 x - \frac{u^2}{2} - \ln|\cos x| + c$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + c$

b We integrate by parts with $u = \ln x \qquad v' = x^{-2}$

$$u' = \frac{1}{x} \qquad v = \frac{x^{-1}}{-1}$$

$$\begin{aligned}\therefore \int \frac{1}{x^2} \ln x \, dx &= -\frac{1}{x}\ln x - \int -\frac{1}{x}\left(\frac{1}{x}\right) dx \\ &= -\frac{\ln x}{x} + \int x^{-2}\, dx \\ &= -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + c \\ &= -\frac{\ln x}{x} - \frac{1}{x} + c\end{aligned}$$

11 Since it is a probability distribution,

$$\sum P(X = x) = \sum_{x=0}^{\infty} a(\tfrac{1}{7})^x = 1$$

But $\sum\limits_{x=0}^{\infty} (\frac{1}{7})^x = \frac{u_1}{1-r} = \frac{1}{1-\frac{1}{7}} = \frac{7}{6}$

So, since $\sum\limits_{x=0}^{\infty} a(\frac{1}{7})^x = 1$, $a \times \frac{7}{6} = 1$ and so $a = \frac{6}{7}$.

12 **a**

$$\begin{aligned} f(x) &= x^3 + 3x^2 + bx + 4 \\ \therefore\ f'(x) &= 3x^2 + 6x + b \\ \therefore\ f''(x) &= 6x + 6 = 6(x+1)\end{aligned}$$

So, $f''(x) = 0$ when $x = -1$.

Now $f(-1) = -1 + 3 - b + 4 = 6 - b$

$\therefore$ $(-1,\ 6-b)$ is a point of inflection.

b A stationary point occurs when $f'(x) = 0$,

$f'(-1) = 0$ in this case. $\therefore$ $3 - 6 + b = 0$

$$\therefore\ b = 3$$

c No stationary points exist when $3x^2 + 6x + b$ has no real roots. $\Delta < 0$ when $36 - 4(3)(b) < 0$

$$\begin{aligned}\therefore\ 12b &> 36 \\ \therefore\ b &> 3\end{aligned}$$

13 **a**

$$\begin{aligned} &(f \circ g)(-4) \\ &= f\,(g(-4)) \\ &= f\,(3(2-(-4))) \\ &= f(18) \\ &= 2(18) - 3 \\ &= 33\end{aligned}$$

b f is $y = 2x - 3$

so f^{-1} is $x = 2y - 3$

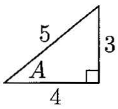
$$\begin{aligned}\therefore\ x + 3 &= 2y \\ \therefore\ y &= \frac{x+3}{2} \\ \therefore\ f^{-1}(x) &= \frac{x+3}{2} \\ \therefore\ f^{-1}(2) &= \frac{2+3}{2} = \frac{5}{2}\end{aligned}$$

14 Probability of not obtaining a score of 35 or more is $\frac{1}{5}$.

If X is the number out of 9 who do not obtain a score of 35 or more, $X \sim B(9, \frac{1}{5})$.

$$P(X = 3) = \binom{9}{3}\left(\tfrac{1}{5}\right)^3\left(\tfrac{4}{5}\right)^6 \approx 0.176.$$

15

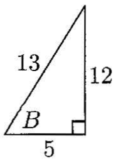


$\therefore$ $\tan A = \frac{3}{4}$

Now $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\therefore\ \frac{-63}{16} = \frac{\frac{3}{4} + t}{1 - \frac{3}{4}t} \qquad \{\text{letting } t = \tan B\}$$

$$\begin{aligned}\therefore\ \frac{-63}{16} &= \frac{3+4t}{4-3t} \\ \therefore\ -252 + 189t &= 48 + 64t \\ \therefore\ 125t &= 300 \\ \therefore\ t &= \tfrac{300}{125} = \tfrac{12}{5}\end{aligned}$$



$\therefore$ $\cos B = \frac{5}{13}$

16 P_n is "$\sum\limits_{r=1}^{n} (r^2 + r) = \dfrac{n(n+1)(n+2)}{3}$" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1^2 + 1 = 2$, RHS $= \dfrac{(1)(2)(3)}{3} = 2$

$\therefore$ P_1 is true.

(2) If P_k is true,

$$(1^2+1) + (2^2+2) + ... + (k^2+k) = \frac{k(k+1)(k+2)}{3}$$

Now $(1^2+1) + (2^2+2) + ... + (k^2+k)$
$+\left((k+1)^2 + (k+1)\right)$

$$\begin{aligned} &= \frac{k(k+1)(k+2)}{3} + (k+1)^2 + k + 1 \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)^2 + 3(k+1)}{3} \\ &= \frac{(k+1)\left[k(k+2) + 3(k+1) + 3\right]}{3} \\ &= \frac{(k+1)(k^2 + 2k + 3k + 6)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)([k+1]+1)([k+1]+2)}{3}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$

{Principle of mathematical induction}

17 If R is perpendicular to both P and Q, the normal $\mathbf{n}$ of R is perpendicular to the normal of P and Q.

$$\therefore\ \mathbf{n} \text{ is parallel to } \begin{pmatrix}6\\-2\\3\end{pmatrix} \times \begin{pmatrix}1\\-3\\5\end{pmatrix}$$

$$\begin{aligned} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & -2 & 3\\ 1 & -3 & 5\end{vmatrix} \\ &= \begin{vmatrix}-2 & 3\\ -3 & 5\end{vmatrix}\mathbf{i} - \begin{vmatrix}6 & 3\\ 1 & 5\end{vmatrix}\mathbf{j} + \begin{vmatrix}6 & -2\\ 1 & -3\end{vmatrix}\mathbf{k} \\ &= (-10+9)\mathbf{i} - (30-3)\mathbf{j} + (-18+2)\mathbf{k} \\ &= -\mathbf{i} - 27\mathbf{j} - 16\mathbf{k}\end{aligned}$$

An equation of R is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \bullet \begin{pmatrix}-1\\-27\\-16\end{pmatrix} = \begin{pmatrix}-1\\-27\\-16\end{pmatrix} \bullet \begin{pmatrix}2\\2\\3\end{pmatrix}$$

$$\begin{aligned}\therefore\ -x - 27y - 16z &= -104 \\ \therefore\ x + 27y + 16z &= 104\end{aligned}$$

18 **a** Since $P(X \geqslant 1) = 0.01$,

$$P(X = 0) = 1 - 0.01 = 0.99$$

But $P(X = 0) = \dfrac{m^0 e^{-m}}{0!} = e^{-m}$

where m is the expected number of errors per page

$$\begin{aligned}\therefore\ e^{-m} &= 0.99 \\ \therefore\ m &= -\ln(0.99) \\ &\approx 0.0101\end{aligned}$$

b $P(X = 1) = \dfrac{m^1 e^{-m}}{1!} = 0.01e^{-0.01} \approx 0.009\,90$

We expect $750 \times 0.009\,90 \approx 7.4$ pages with exactly 1 mistake.

19 If $z = r\,\text{cis}\,\theta, \quad 1 < r < 2$

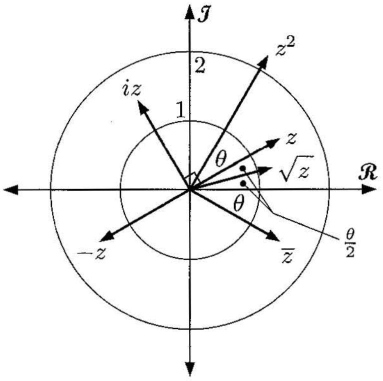
a $-z = -r\,\text{cis}\,\theta = r\,\text{cis}\,(\theta + \pi)$

b $\overline{z} = r\,\text{cis}\,(-\theta)$

c $iz = r\,\text{cis}\,(\theta + \frac{\pi}{2})$

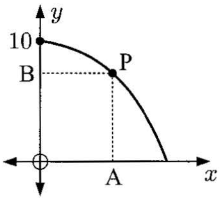
d $z^2 = r^2\,\text{cis}\,(2\theta),$
$1 < r^2 < 4$

e $\sqrt{z} = \sqrt{r}\,\text{cis}\left(\frac{\theta}{2}\right),$
$1 < \sqrt{r} < \sqrt{2}$



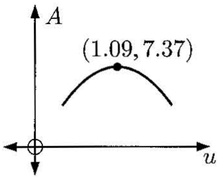
20 Let X be the score for the mathematics test, so
$X \sim \text{N}(64,\ (8.352)^2)$.
$\text{P}(X \geqslant 80) \approx 0.0277$

21 $y = 10 - xe^x$
Let P be at $(u,\ 10 - ue^u)$
and let A be the area of OAPB.
$\therefore \quad A = u(10 - ue^u)$
$= 10u - u^2e^u$

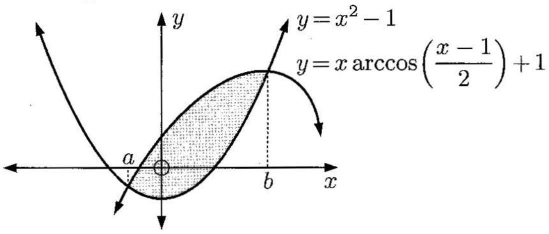


Using technology, the graph of A against u is:

$\therefore$ maximum area ≈ 7.37 units^2.



22 Using technology:



$$\text{Area} = \int_a^b \left(x\arccos\left(\frac{x-1}{2}\right) + 1 - x^2 + 1\right) dx$$

where $a \approx -0.6330, \quad b \approx 2.0227$
$\therefore$ area ≈ 4.84 units^2

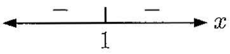
23 $f(x) = \dfrac{x^2 + 1}{e^x}$

a As $x \to \infty,\ e^x \to \infty$ more rapidly than $x^2 + 1 \to \infty$
$\therefore$ as $x \to \infty,\ f(x) \to 0$ (above).

b $$f'(x) = \frac{2xe^x - (x^2+1)e^x}{e^{2x}} = \frac{2x - x^2 - 1}{e^x}$$
$$= \frac{-(x-1)^2}{e^x}$$

c $f'(x) = 0$ when $(x-1)^2 = 0 \quad \therefore \quad x = 1$

Sign diagram of $f'(x)$:

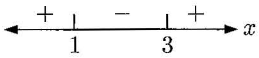


$\therefore$ there is a stationary inflection point at $(1,\ f(1))$, which is $\left(1,\ \dfrac{2}{e}\right)$.

d $$f''(x) = \frac{(2-2x)e^x - (2x - x^2 - 1)e^x}{e^{2x}}$$
$$= \frac{2 - 2x - 2x + x^2 + 1}{e^x}$$
$$= \frac{x^2 - 4x + 3}{e^x}$$

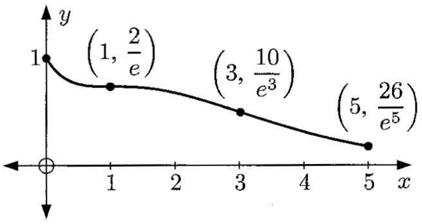
e $f''(x) = 0$ when $x^2 - 4x + 3 = 0$
$\therefore \quad (x-3)(x-1) = 0$
$\therefore \quad x = 3$ or 1

Sign diagram of $f''(x)$:



$\therefore$ there is a non-stationary inflection at $\left(3,\ \dfrac{10}{e^3}\right)$

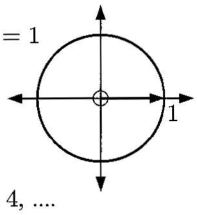
f $f(0) = 1$ and $f(5) = \dfrac{26}{e^5} \approx 0.175$



24 **a** $x = 6 + 2\cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right)$

Tide is highest when $\cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right) = 1$
$\therefore \quad \frac{4\pi}{25}t + \frac{\pi}{3} = k2\pi$
$\therefore \quad \frac{4\pi}{25}t = -\frac{\pi}{3} + k2\pi$
$\therefore \quad 4\pi t = -\frac{25\pi}{3} + k50\pi$
$\therefore \quad t = -\frac{25}{12} + k\frac{25}{2}, \quad k = 0, 1, 2, 3, 4, ...$

When $k = 1,\ t = -\frac{25}{12} + \frac{25}{2} = \frac{125}{12}$
$\therefore \quad t = 10$ hours 25 minutes
$\therefore$ the first high tide is at 10:25 pm September 1.

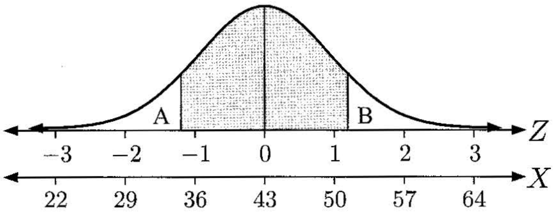


b period $= \dfrac{2\pi}{\frac{4\pi}{25}} = \frac{25}{2}$
$\therefore$ there is $12\frac{1}{2}$ hours between high tides.

c If $x = 5.5, \quad 6 + 2\cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right) = 5.5$
$\therefore \quad 2\cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right) = -0.5$
$\therefore \quad \cos\left(\frac{4\pi}{25}t + \frac{\pi}{3}\right) = -0.25$

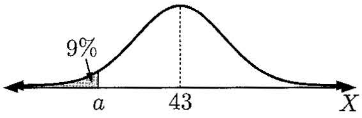
3:00 pm on September 2 is $t = 27$
So, we seek the solution when t is first > 27.
Using technology, $t \approx 31.788$
So, the first time is 31 h 47 min after 12:00 midday September 1. This is 7:47 pm on September 2.

25 **a**



b Let a be the greatest age at death in region A

$\therefore \ \mathrm{P}(X \leqslant a) = 0.09$



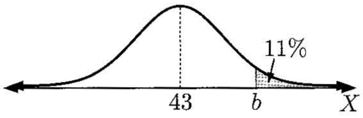
$\therefore \ a \approx 33.6$

The greatest age is about 33 years and 7 months.

c Let b be the lowest age at death in region B

$\therefore \ \mathrm{P}(X \geqslant b) = 0.11$

$\therefore \ \mathrm{P}(X \leqslant b) = 0.89$



$\therefore \ b \approx 51.6$

So, the lowest age is about 51 years and 7 months.

d Since the total area is 1, the shaded region has area $1 - (0.09 + 0.11) = 0.80$.

SOLUTIONS TO EXAMINATION PRACTICE SET 11

1 **a** $-8 = 8(-1) = 8 \text{ cis } \pi$

b $z^3 = -8$

$\therefore \ z^3 = 8 \operatorname{cis}(\pi + k2\pi), \quad k \in \mathbb{Z}$

$\therefore \ z = 2 \operatorname{cis}\left(\dfrac{\pi + k2\pi}{3}\right)$ {De Moivre}

$\therefore \ z = 2 \operatorname{cis} \frac{\pi}{3}, \quad 2 \operatorname{cis} \pi, \quad 2 \operatorname{cis}\left(\frac{5\pi}{3}\right) \qquad \{k = 0, 1, 2\}$

$\therefore \ z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \quad 2(-1), \quad 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\therefore \ z = 1 + i\sqrt{3}, \quad -2, \quad 1 - i\sqrt{3}$

2 f is $y = \sqrt{5 - 2x}, \ x \leqslant \frac{5}{2}$

so f^{-1} is $x = \sqrt{5 - 2y}$

$\therefore \ x^2 = 5 - 2y$

$\therefore \ y = \dfrac{5 - x^2}{2}$

So, $f^{-1}(5) = \dfrac{5 - (5^2)}{2} = -10$

3 Stretching $f(x)$ vertically by a factor of 2 gives $2f(x)$.

Compressing $2f(x)$ horizontally by a factor of 2 gives $2f(2x)$.

A translation of $\frac{1}{2}$ horizontally gives $2f\left(2(x - \frac{1}{2})\right)$
$= 2f(2x - 1)$.

A translation of -3 vertically gives $2f(2x - 1) - 3$.

So, since $f(x) = x + 2, \quad 2f(2x - 1) - 3$

$= 2\left[(2x - 1) + 2\right] - 3$

$= 2(2x + 1) - 3$

$= 4x - 1$

4 **a** $f(x) = 2x^3 - 9x^2 + 30x - 13$

$\therefore \ f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 30\left(\frac{1}{2}\right) - 13 = 0$

b So, $2x - 1$ is a factor of $f(x)$

$\therefore \ f(x) = (2x - 1)(x^2 + ax + 13)$

Equating coefficients of x^2, $\quad -9 = 2a - 1$

$\therefore \ a = -4$

so $f(x) = (2x - 1)(x^2 - 4x + 13)$

Now $x^2 - 4x + 13 = 0$

if $x = \dfrac{4 \pm \sqrt{16 - 52}}{2} = \dfrac{4 \pm 6i}{2}$

$\therefore \ x = 2 + 3i$ or $2 - 3i$

$\therefore \ 2x^3 - 9x^2 + 30x - 13$

$= (2x - 1)\,(x - (2 + 3i))\,(x - (2 - 3i))$

5 $\tan\left(A + \frac{\pi}{4}\right) = 3 \quad \therefore \ \dfrac{\tan A + \tan\frac{\pi}{4}}{1 - \tan A \tan\frac{\pi}{4}} = 3$

$\therefore \ \dfrac{\tan A + 1}{1 - \tan A} = 3$

$\therefore \ \tan A + 1 = 3 - 3\tan A$

$\therefore \ 4\tan A = 2$

$\therefore \ \tan A = \frac{1}{2}$

6 $2^{x+1} = 5^{x-1}$

$\therefore \ \log 2^{x+1} = \log 5^{x-1}$

$\therefore \ (x + 1)\log 2 = (x - 1)\log 5$

$\therefore \ x + 1 = (x - 1)\dfrac{\log 5}{\log 2}$

$\therefore \ x + 1 = (x - 1)\log_2 5$

$\therefore \ x + 1 = x\log_2 5 - \log_2 5$

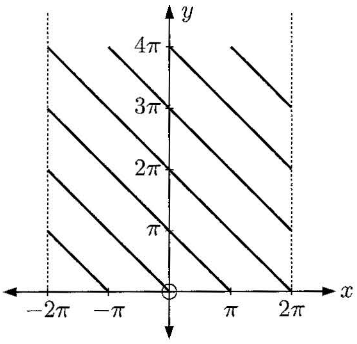
$\therefore \ x(1 - \log_2 5) = -1 - \log_2 5$

$\therefore \ x = \dfrac{1 + \log_2 5}{\log_2 5 - 1}$

7 $\sin(x + y) = 0$

$\therefore \ x + y = k\pi, \quad k \in \mathbb{Z}$

which is a family of straight lines with gradient -1.



8 $\int_{-a}^{a} \left(3x^2 - 8x + 2\right) dx = 12a$

$\therefore \ \left[\dfrac{3x^3}{3} - \dfrac{8x^2}{2} + 2x\right]_{-a}^{a} = 12a$

$\therefore \ (a^3 - 4a^2 + 2a) - (-a^3 - 4a^2 - 2a) = 12a$

$\therefore \ 2a^3 + 4a = 12a$

$\therefore \ 2a^3 - 8a = 0$

$\therefore \ 2a(a^2 - 4) = 0$

$\therefore \ 2a(a + 2)(a - 2) = 0$

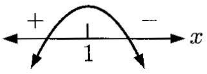
$\therefore \ a = 0$ or ± 2

$\therefore \ a = 2 \quad \{a > 0\}$

9 **a** $f(x) = 4xe^{-x}, \quad x \geqslant 0$

$$\therefore \quad f'(x) = 4e^{-x} + 4xe^{-x}(-1)$$
$$= 4e^{-x}(1-x)$$
$$\therefore \quad f'(x) = 0 \quad \text{when} \quad x = 1 \quad \{\text{as } e^{-x} > 0 \text{ for all } x\}$$

$f'(x)$ has sign diagram:

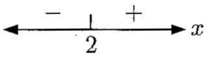


$\therefore$ there is a local maximum at $\left(1, \dfrac{4}{e}\right)$.

b $f''(x) = -4e^{-x}(1-x) + 4e^{-x}(-1)$

$$= 4e^{-x}(-1+x-1)$$
$$= 4e^{-x}(x-2)$$
$$\therefore \quad f''(x) = 0 \quad \text{when} \quad x = 2$$

$f''(x)$ has sign diagram:



As the signs alternate, there is a point of inflection at $\left(2, \dfrac{8}{e^2}\right)$.

10 $1 - \tan^2 x + \tan^4 x - \ldots.$ is geometric with $u_1 = 1$ and $r = -\tan^2 x$.

a It converges when $-1 < r < 1$

$$\therefore \quad -1 < -\tan^2 x < 1$$
$$\therefore \quad -1 < \tan^2 x < 1$$
$$\therefore \quad 0 \leqslant \tan^2 x < 1 \qquad \{\text{as } a^2 \geqslant 0 \text{ for all real } a\}$$
$$\therefore \quad -1 < \tan x < 1$$
$$\therefore \quad x \in [0, \tfrac{\pi}{4}[\text{ or }]\tfrac{3\pi}{4}, \tfrac{5\pi}{4}[\text{ or }]\tfrac{7\pi}{4}, 2\pi].$$

b $S = \dfrac{u_1}{1-r} = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{\sec^2 x} = \cos^2 x$

c When $S = \frac{1}{2}$, $\cos^2 x = \frac{1}{2}$

$$\therefore \quad \cos x = \pm\tfrac{1}{\sqrt{2}}$$
$$\therefore \quad x = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4}$$

However, none of these are in the convergent range, so the sum is never $\frac{1}{2}$.

11 In augmented form, we use row reduction:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ 4 & -1 & -1 & k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & k-4 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & k-4 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

a The system has an infinite number of solutions if $k = 4$.

b The system has no solutions if $k \neq 4$.

12 P_n is "$2^{n-1} \leqslant n!$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $2^{n-1} = 2^0 = 1$ and $1! = 1$ as $1 \leqslant 1$ is a true statement, P_1 is true.

(2) Suppose P_k is true, so $2^{k-1} \leqslant k!$ or $k! \geqslant 2^{k-1}$.

Now $(k+1)! - 2^k$

$$= (k+1)k! - 2^k$$
$$\geqslant (k+1)2^{k-1} - 2^k$$
$$\geqslant 2^{k-1}(k+1-2)$$
$$\geqslant 2^{k-1}(k-1)$$
$$\geqslant 0 \quad \text{as} \quad 2^{k-1} > 0 \quad \text{and} \quad k \geqslant 1.$$

This means $(k+1)! \geqslant 2^k$ or $2^k \leqslant (k+1)!$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$
{Principle of mathematical induction}

13 **a** Since $\mathrm{P}(N) = \frac{3}{5}$, $\mathrm{P}(N') = 1 - \frac{3}{5} = \frac{2}{5}$

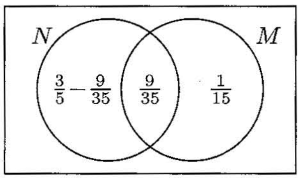
b Since $\mathrm{P}(M \mid N) = \dfrac{\mathrm{P}(M \cap N)}{\mathrm{P}(N)} = \frac{3}{7}$

and $\mathrm{P}(N) = \frac{3}{5}$, $\mathrm{P}(M \cap N) = \frac{3}{5} \times \frac{3}{7} = \frac{9}{35}$.

Since $\mathrm{P}(M \mid N') = \dfrac{\mathrm{P}(M \cap N')}{\mathrm{P}(N')} = \frac{1}{6}$

and $\mathrm{P}(N') = \frac{2}{5}$, then $\mathrm{P}(M \cap N') = \frac{2}{5} \times \frac{1}{6} = \frac{1}{15}$.

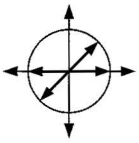
$\mathrm{P}(M \cup N) = \frac{3}{5} + \frac{1}{15} = \frac{2}{3}$



$\therefore$ $\mathrm{P}[(M \cup N)'] = \mathrm{P}(M' \cap N') = 1 - \frac{2}{3} = \frac{1}{3}$.

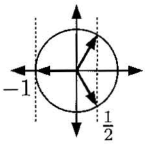
14 **a** $\sin 2x + \cos 2x = 1, \quad 0 \leqslant x \leqslant 2\pi$

$$\therefore \quad 2\sin x\cos x + 1 - 2\sin^2 x = 1$$
$$\therefore \quad 2\sin x\cos x - 2\sin^2 x = 0$$
$$\therefore \quad 2\sin x(\cos x - \sin x) = 0$$
$$\therefore \quad \sin x = 0 \quad \text{or} \quad \cos x = \sin x$$
$$\therefore \quad \sin x = 0 \quad \text{or} \quad \tan x = 1$$
$$\therefore \quad x = 0, \tfrac{\pi}{4}, \pi, \tfrac{5\pi}{4}, 2\pi$$



b $\cos 2x = -\cos x, \quad 0 \leqslant x \leqslant 2\pi$

$$\therefore \quad 2\cos^2 x - 1 + \cos x = 0$$
$$\therefore \quad (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \quad \cos x = \tfrac{1}{2} \text{ or } -1$$
$$\therefore \quad x = \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}$$



15 Let $u = 1 - x$, $\dfrac{du}{dx} = -1$.

$$\therefore \int \frac{x^2}{(1-x)^3}\,dx = -\int \frac{(1-u)^2}{u^3}\frac{du}{dx}\,dx$$
$$= -\int \frac{1-2u+u^2}{u^3}\,du$$
$$= -\int \left(u^{-3} - 2u^{-2} + \frac{1}{u}\right) du$$
$$= -\frac{u^{-2}}{-2} + \frac{2u^{-1}}{-1} - \ln|u| + c$$
$$= \frac{1}{2(1-x)^2} - \frac{2}{1-x} - \ln|1-x| + c$$

16 **a** Let $y = \dfrac{x}{\sqrt{x-6}} = \dfrac{x}{(x-6)^{\frac{1}{2}}}$

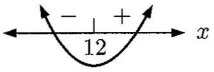
$$\therefore \quad \frac{dy}{dx} = \frac{1(x-6)^{\frac{1}{2}} - x\left(\frac{1}{2}\right)(x-6)^{-\frac{1}{2}}}{(x-6)^1}$$

$$= \frac{\dfrac{\sqrt{x-6}}{1} - \dfrac{x}{2\sqrt{x-6}}}{(x-6)}$$

$$= \frac{2(x-6)-x}{2(x-6)^{\frac{3}{2}}} = \frac{x-12}{2(x-6)^{\frac{3}{2}}}$$

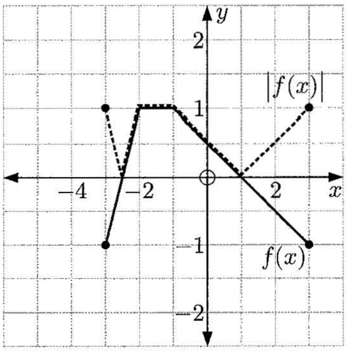
b $\dfrac{dy}{dx} = 0$ when $x - 12 = 0 \quad \therefore \quad x = 12$

Sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore$ x-coordinate of local minimum is 12.

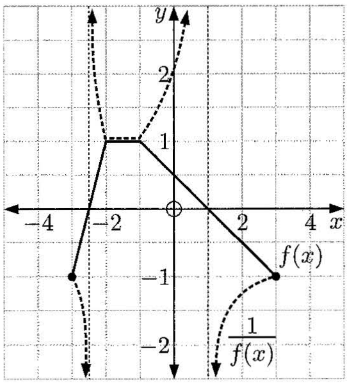
17 **a**



b The y-intercept of $y = f(x)$ is $f(0) = \frac{1}{2}$.

So, the y-intercept of $y = \dfrac{1}{f(x)}$ is $\dfrac{1}{f(0)} = 2$.

c



18 $a + \dfrac{a}{b^2} + \dfrac{a}{b^4} + \ldots.$ is geometric with $u_1 = a$ and $r = \dfrac{1}{b^2}$.

a It converges when $-1 < r < 1 \quad \therefore \quad -1 < \dfrac{1}{b^2} < 1$

But $b^2 > 0$ for all b $\quad \therefore \quad 0 < \dfrac{1}{b^2} < 1$

$\therefore \quad b^2 > 1$

$\therefore \quad b < -1$ or $b > 1$

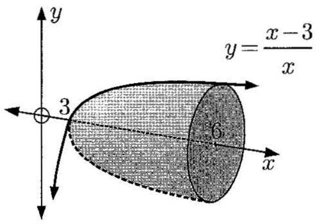
b $S = \dfrac{u_1}{1-r} = \dfrac{a}{1 - \dfrac{1}{b^2}} \times \dfrac{b^2}{b^2} = \dfrac{ab^2}{b^2 - 1}$

c $0.\overline{32} = \frac{32}{10^2} + \frac{32}{10^4} + \frac{32}{10^6} + \ldots.$

$= \frac{32}{10^2}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots.\right)$

$= \frac{32}{100}\left(\dfrac{1 \times 10^2}{10^2 - 1}\right) = \frac{32}{99}$

19 Volume



$$= \pi \int_3^6 y^2\,dx$$

$$= \pi \int_3^6 \left(1 - \frac{3}{x}\right)^2 dx$$

$$= \pi \int_3^6 \left(1 - \frac{6}{x} + 9x^{-2}\right) dx$$

$$= \pi \left[x - 6\ln|x| + \frac{9x^{-1}}{-1}\right]_3^6$$

$$= \pi\left((6 - 6\ln 6 - \tfrac{3}{2}) - (3 - 6\ln 3 - 3)\right)$$

$$= \pi\left(\tfrac{9}{2} - 6\ln 6 + 6\ln 3\right)$$

$$= \pi\left(\tfrac{9}{2} + 6\ln(\tfrac{1}{2})\right)$$

$$= \pi\left(\tfrac{9}{2} - 6\ln 2\right) \text{ units}^3$$

20 $\cos\theta + i\sin\theta = \operatorname{cis}\theta$
$\cos\theta - i\sin\theta = \operatorname{cis}(-\theta)$

$$\therefore \quad \frac{\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^5 \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^7}$$

$$= \frac{\left[\operatorname{cis}\left(-\frac{\pi}{3}\right)\right]^5 \times \left[\operatorname{cis}\frac{\pi}{4}\right]^3}{\left[\operatorname{cis}\left(-\frac{\pi}{12}\right)\right]^7}$$

$$= \frac{\operatorname{cis}\left(-\frac{5\pi}{3}\right) \times \operatorname{cis}\left(\frac{3\pi}{4}\right)}{\operatorname{cis}\left(-\frac{7\pi}{12}\right)} \quad \text{\{De Moivre\}}$$

$$= \operatorname{cis}\left(-\tfrac{5\pi}{3} + \tfrac{3\pi}{4} + \tfrac{7\pi}{12}\right)$$

$$= \operatorname{cis}\left(-\tfrac{\pi}{3}\right)$$

$$= \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

21 **a** $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 2 & -1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -3 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-3-1)\mathbf{i} - (1+2)\mathbf{j} + (-1+6)\mathbf{k}$$

$$= -4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

b $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -4 & -3 & 5 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 3 \\ -3 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -4 & -3 \end{vmatrix}\mathbf{k}$$

$$= (-5+9)\mathbf{i} - (10+12)\mathbf{j} + (-6-4)\mathbf{k}$$

$$= 4\mathbf{i} - 22\mathbf{j} - 10\mathbf{k}$$

$$\mathbf{b}(\mathbf{a} \bullet \mathbf{c}) = (\mathbf{i} - 3\mathbf{j} - \mathbf{k})(4 + 1 + 3)$$

$$= 8\mathbf{i} - 24\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{c}(\mathbf{a} \bullet \mathbf{b}) = (2\mathbf{i} - \mathbf{j} + \mathbf{k})(2 + 3 - 3)$$

$$= 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

and $\mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b}) = 4\mathbf{i} - 22\mathbf{j} - 10\mathbf{k}$
$= \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

22 **a** $w^6 = 64 = 64\operatorname{cis}(0 + k2\pi), \quad k \in \mathbb{Z}$

$\therefore \quad w^6 = 2^6\operatorname{cis}(k2\pi), \quad k \in \mathbb{Z}$

$\therefore \quad w = 2\operatorname{cis}\left(\frac{k2\pi}{6}\right), \quad k \in \mathbb{Z}$ {De Moivre}

$$\therefore\ w = 2\operatorname{cis} 0,\ \ 2\operatorname{cis}\left(\tfrac{\pi}{3}\right),\ \ 2\operatorname{cis}\left(\tfrac{2\pi}{3}\right),\ \ 2\operatorname{cis}\left(\tfrac{3\pi}{3}\right),\ \ 2\operatorname{cis}\left(\tfrac{4\pi}{3}\right),\ \ 2\operatorname{cis}\left(\tfrac{5\pi}{3}\right)$$

$$\therefore\ w = \pm 2,\ \ 2\left(\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}i\right),\ \ 2\left(-\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}i\right)$$

$$\therefore\ w = \pm 2,\ \ 1 \pm i\sqrt{3},\ \ -1 \pm i\sqrt{3}$$

b If $(z+1)^6 = 64(z-2)^6$, then $\left(\dfrac{z+1}{z-2}\right)^6 = 64$

$$\therefore\ \frac{z+1}{z-2} = \pm 2,\ \ \frac{z+1}{z-2} = 1 \pm i\sqrt{3},$$

or $\dfrac{z+1}{z-2} = -1 \pm i\sqrt{3}$

Suppose $\dfrac{z+1}{z-2} = w$

$$\therefore\ z + 1 = wz - 2w$$

$$\therefore\ z(1-w) = -2w - 1$$

$$\therefore\ z = \frac{2w+1}{w-1}$$

If $\dfrac{z+1}{z-2} = 2,\ \ z = \frac{5}{1} = 5$

If $\dfrac{z+1}{z-2} = -2,\ \ z = \frac{-3}{-3} = 1$

If $\dfrac{z+1}{z-2} = 1 + i\sqrt{3},\ \ z = \dfrac{2 + 2\sqrt{3}i + 1}{i\sqrt{3}}$

which simplifies to $2 - i\sqrt{3}$.

If $\dfrac{z+1}{z-2} = 1 - i\sqrt{3},\ \ z = \dfrac{2 - 2\sqrt{3}i + 1}{-i\sqrt{3}}$

which simplifies to $2 + i\sqrt{3}$.

If $\dfrac{z+1}{z-2} = -1 + i\sqrt{3},\ \ z = \dfrac{-2 + 2\sqrt{3}i + 1}{i\sqrt{3} - 2}$

which simplifies to $\frac{1}{7}(8 - 3\sqrt{3}i)$.

If $\dfrac{z+1}{z-2} = -1 - i\sqrt{3},\ \ z = \dfrac{-2 - 2\sqrt{3}i + 1}{-i\sqrt{3} - 2}$

which simplifies to $\frac{1}{7}(8 + 3\sqrt{3}i)$.

$$\therefore\ z = 5,\ 1,\ 2 \pm i\sqrt{3},\ \tfrac{1}{7}(8 \pm 3\sqrt{3}i).$$

23 P_n is "If $u_{n+2} = u_n + u_{n+1},\ u_1 = u_2 = 1$

then $u_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$", $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$,

$$u_1 = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \text{ which is true.}$$

$\therefore$ P_1 is true.

If $n = 2$,

$$u_2 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{(1+2\sqrt{5}+5)-(1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 \text{ which is true.}$$

$\therefore$ P_2 is true.

(2) Now suppose P_k and P_{k+1} are true.

So, $u_k = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$ and

$$u_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

Now $u_{k+2} = u_{k+1} + u_k$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left[\frac{1+\sqrt{5}}{2}+1\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left[\frac{1-\sqrt{5}}{2}+1\right]}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

We notice that

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

and likewise $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$.

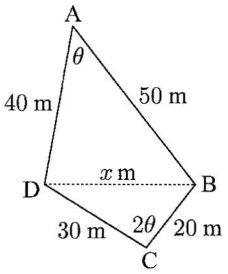
So,

$$u_{k+2} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^{2}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}}$$

Thus P_{k+2} is true whenever P_k and P_{k+1} are true and P_1 and P_2 are true.

$\therefore$ P_n is true for all $n \in \mathbb{Z}^+$
{Principle of mathematical induction}

24 **a**



By the cosine rule:

$$x^2 = 40^2 + 50^2 - 2 \times 40 \times 50 \cos\theta$$

and $x^2 = 30^2 + 20^2 - 2 \times 30 \times 20 \cos 2\theta$

$$\therefore\ 1600 + 2500 - 4000\cos\theta = 900 + 400 - 1200\cos 2\theta$$

$$\therefore\ 4100 - 4000\cos\theta = 1300 - 1200\cos 2\theta$$

$$\therefore\ 41 - 40\cos\theta = 13 - 12\cos 2\theta$$

$$\therefore\ 3\cos 2\theta - 10\cos\theta + 7 = 0$$

$$\therefore\ 3(2\cos^2\theta - 1) - 10\cos\theta + 7 = 0$$

$$\therefore\ 6\cos^2\theta - 3 - 10\cos\theta + 7 = 0$$

$$\therefore\ 6\cos^2\theta - 10\cos\theta + 4 = 0$$

$$\therefore\ 3\cos^2\theta - 5\cos\theta + 2 = 0$$

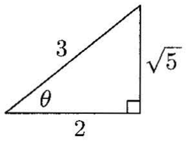
$$\therefore\ (3\cos\theta - 2)(\cos\theta - 1) = 0$$

$$\therefore\ \cos\theta = \tfrac{2}{3} \text{ or } 1$$

Now $\cos\theta = 1$ is impossible, so $\cos\theta = \frac{2}{3}$.

b $\cos\theta = \frac{2}{3}$ and $\sin\theta = \frac{\sqrt{5}}{3}$

$$\therefore\ \sin 2\theta = 2\sin\theta\cos\theta = 2 \times \tfrac{\sqrt{5}}{3} \times \tfrac{2}{3} = \tfrac{4\sqrt{5}}{9}$$



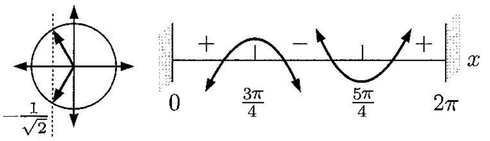
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}(40 \times 50 \times \sin\theta) + \tfrac{1}{2}(30 \times 20 \times \sin 2\theta)\\ &= 1000\sin\theta + 300\sin 2\theta\\ &= 1000\left(\tfrac{\sqrt{5}}{3}\right) + 300\left(\tfrac{4\sqrt{5}}{9}\right)\\ &= 1000\left(\tfrac{\sqrt{5}}{3}\right) + 100\left(\tfrac{4\sqrt{5}}{3}\right)\\ &= \tfrac{\sqrt{5}}{3}(1000+400)\\ &= \tfrac{1400}{3}\sqrt{5}\ \text{m}^2\end{aligned}$$

25 $f(x) = \dfrac{\sin x}{\cos x + \sqrt{2}}, \quad 0 < x < 2\pi$

a **i** $$\begin{aligned}f'(x) &= \frac{\cos x(\cos x + \sqrt{2}) - \sin x(-\sin x)}{(\cos x + \sqrt{2})^2}\\ &= \frac{\cos^2 x + \sqrt{2}\cos x + \sin^2 x}{(\cos x + \sqrt{2})^2}\\ &= \frac{1 + \sqrt{2}\cos x}{(\cos x + \sqrt{2})^2}\end{aligned}$$

ii When $f'(x) = 0, \quad 1 + \sqrt{2}\cos x = 0$

$\therefore \quad \cos x = -\frac{1}{\sqrt{2}}$ and so $x = \frac{3\pi}{4}, \frac{5\pi}{4}$



$\therefore$ local maximum at $\left(\frac{3\pi}{4}, 1\right)$,
local minimum at $\left(\frac{5\pi}{4}, -1\right)$.

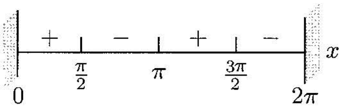
b **i** $$\begin{aligned}f''(x) &= \frac{\left(\begin{array}{r}(-\sqrt{2}\sin x)(\cos x + \sqrt{2})^2\\ -(1+\sqrt{2}\cos x)2(\cos x + \sqrt{2})(-\sin x)\end{array}\right)}{(\cos x + \sqrt{2})^4}\\ &= \frac{-\sqrt{2}\sin x\cos x - 2\sin x + 2\sin x + 2\sqrt{2}\sin x\cos x}{(\cos x + \sqrt{2})^3}\\ &= \frac{\sqrt{2}\sin x\cos x}{(\cos x + \sqrt{2})^3}\end{aligned}$$

ii When $f''(x) = 0, \quad \sin x\cos x = 0$

$\therefore \quad \sin x = 0$ or $\cos x = 0$

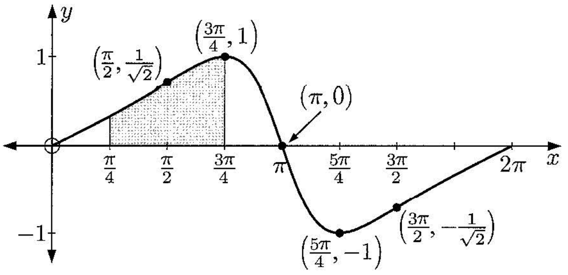
$\therefore \quad x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

There are changes in signs of $f''(x)$:



$\therefore$ there are points of inflection at $\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, -\frac{1}{\sqrt{2}}\right)$.

c



d $$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x}{\cos x + \sqrt{2}}\,dx\\ &= \left[-\ln|\cos x + \sqrt{2}|\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\\ &= -\ln\left|-\tfrac{1}{\sqrt{2}} + \sqrt{2}\right| + \ln\left|\tfrac{1}{\sqrt{2}} + \sqrt{2}\right|\\ &= -\ln\left|\sqrt{2} - \tfrac{1}{2}\sqrt{2}\right| + \ln\left|\sqrt{2} + \tfrac{1}{2}\sqrt{2}\right|\\ &= -\ln(\tfrac{1}{2}\sqrt{2}) + \ln(\tfrac{3}{2}\sqrt{2})\\ &= \ln\left(\frac{\frac{3}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}}\right)\\ &= \ln 3 \ \text{units}^2\end{aligned}$$

SOLUTIONS TO EXAMINATION PRACTICE SET 12

1 For $(2-x)^8, \quad T_{r+1} = \binom{8}{r} 2^{8-r}(-x)^r$
$= \binom{8}{r} 2^{8-r}(-1)^r x^r$

For $(1+2x)(2-x)^8$, the coefficient of x^7

$= 1 \times \binom{8}{7} 2^{8-7}(-1)^7 + 2\binom{8}{6} 2^{8-6}(-1)^6$

$= 8 \times 2 \times -1 + 2 \times 28 \times 4$

$= 208$

2 Sequence is geometric, $u_1 = 100, \quad r = 1.1$

a $u_n = u_1 r^{n-1} \quad \therefore \quad u_n = 100 \times (1.1)^{n-1}$

b $S_n = \dfrac{u_1(r^n - 1)}{r-1} = \dfrac{100\,((1.1)^n - 1)}{0.1}$

$\therefore \quad S_n = 1000\,((1.1)^n - 1)$

c We want n such that $S_n > 4000$

$\therefore \quad 1000\,((1.1)^n - 1) > 4000$

$\therefore \quad (1.1)^n - 1 > 4$

$\therefore \quad (1.1)^n > 5$

$\therefore \quad \log(1.1)^n > \log 5$

$\therefore \quad n\log(1.1) > \log 5$

$\therefore \quad n > \dfrac{\log 5}{\log 1.1}$

$\therefore \quad n > 16.9$

$\therefore \quad n = 17, 18, 19, 20, ...$

The amount first exceeds $4000 on his 17th birthday.

3 **a** $6 \times 6 \times 6 = 216$ numbers

b $6 \times 5 \times 4 = 120$ numbers

c The first digit is 4, 5, 6, or 7

- if the first digit is 4 or 6

  $\boxed{2}\boxed{4}\boxed{2} = 16$

  $\uparrow$ 4 or 6 $\quad \uparrow$ other 2 evens

- if the first digit is 5 or 7

  $\boxed{2}\boxed{4}\boxed{3} = 24$

  $\uparrow$ 3 possible evens

$\therefore$ total number is 40

d We can choose 3 from 6 in $\binom{6}{3} = 20$ possible ways and there is one order (ascending) for each way.

$\therefore$ total number of ways $= 20$

4 $$\int_0^2 \frac{1}{1+ux}\,dx = \frac{1}{u}$$

$$\therefore \quad \left[\frac{1}{u}\ln|1+ux|\right]_0^2 = \frac{1}{u}$$

$$\therefore \quad \frac{1}{u}\ln|1+2u| - \frac{1}{u}\ln 1 = \frac{1}{u}$$

$$\therefore \quad \ln|1+2u| = 1$$

$$\therefore\ |1+2u| = e$$
$$\therefore\ 1+2u = \pm e$$
$$\therefore\ 2u = \pm e - 1$$
$$\therefore\ u = \frac{e-1}{2} \text{ or } \frac{-e-1}{2}$$

Consider $u = -\left(\frac{e+1}{2}\right)$.

The function $\dfrac{1}{1+ux} = \dfrac{1}{1-\left(\frac{e+1}{2}\right)x}$ is not defined

when $x = \dfrac{2}{e+1}$.

Since $0 < \dfrac{2}{e+1} < 2$, the definite integral

$\displaystyle\int_0^2 \frac{1}{1-\left(\frac{e+1}{2}\right)x}\,dx$ is therefore not defined.

Hence, the only value is $u = \dfrac{e-1}{2}$.

5 **a** The line has direction $\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

We label $(-1, 1, 1)$ on the line as point A, and $(1, 3, -2)$ on the plane as point B.

Then $\overrightarrow{AB} = \begin{pmatrix}2\\2\\-3\end{pmatrix}$ and a normal to the plane is

$$\begin{pmatrix}1\\2\\-2\end{pmatrix} \times \begin{pmatrix}2\\2\\-3\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -2\\ 2 & 2 & -3\end{vmatrix}$$
$$= \begin{vmatrix}2 & -2\\ 2 & -3\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & -2\\ 2 & -3\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}\mathbf{k}$$
$$= (-6+4)\mathbf{i} - (-3+4)\mathbf{j} + (2-4)\mathbf{k}$$
$$= -2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

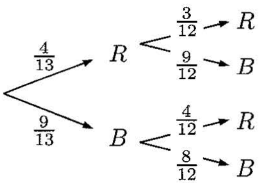
An equation of the plane is

$$\begin{pmatrix}-2\\-1\\-2\end{pmatrix} \bullet \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2\\-1\\-2\end{pmatrix} \bullet \begin{pmatrix}1\\3\\-2\end{pmatrix}$$
$$\therefore\ -2x - y - 2z = -1$$
$$\therefore\ 2x + y + 2z = 1$$

b Distance of plane to $(0, 0, 0)$ is

$$\frac{|2(0)+1(0)+2(0)-1|}{\sqrt{2^2+1^2+2^2}} = \tfrac{1}{3} \text{ unit}$$

6



P(one of each colour)

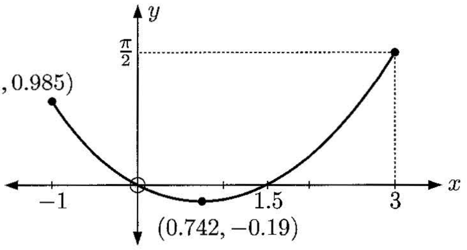
$$= \tfrac{4}{13} \times \tfrac{9}{12} + \tfrac{9}{13} \times \tfrac{4}{12}$$
$$= \tfrac{72}{156}$$
$$\approx 0.462$$

7 $y = x \arcsin\left(\frac{x}{3} - 0.5\right), \quad -1 \leqslant x \leqslant 3$

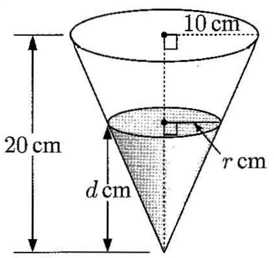
$y(-1) = -1 \arcsin\left(-\frac{5}{6}\right) \approx 0.985$

$y(3) = 3 \arcsin(0.5) = \frac{\pi}{2}$

Using technology, $y_{\min} \approx -0.190$, where $x \approx 0.742$.



8



If the depth of water is d cm

$$\frac{r}{d} = \tfrac{10}{20} = \tfrac{1}{2}$$
$$\therefore\ r = \tfrac{1}{2}d$$

Let the volume of water be V cm^3.

$$\therefore\ V = \tfrac{1}{3}\pi r^2 d$$
$$\therefore\ V = \tfrac{1}{3}\pi\left(\frac{d^2}{4}\right)d = \tfrac{\pi}{12}d^3$$
$$\therefore\ \frac{dV}{dt} = \tfrac{\pi}{12} \times 3d^2\,\frac{dd}{dt}$$

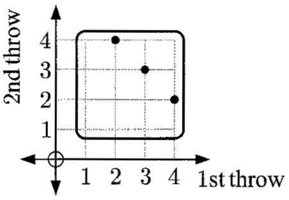
Particular case: $d = 15$, $\dfrac{dV}{dt} = 30 \text{ cm}^3\,\text{s}^{-1}$

$$\therefore\ 30 = \tfrac{\pi}{12} \times 3 \times 15^2 \times \frac{dd}{dt}$$
$$\therefore\ \frac{dd}{dt} = \tfrac{8}{15\pi} \text{ cm s}^{-1}$$

So, the depth is increasing at $\frac{8}{15\pi}$ cm s^{-1}.

9 As the probabilities add to 1, $\quad \frac{2}{7} + \frac{1}{3} + x + \frac{2}{21} = 1$

$$\therefore\ 6 + 7 + 21x + 2 = 21$$
$$\therefore\ 21x = 6$$
$$\therefore\ x = \tfrac{2}{7}$$



A total of 6 after two rolls can only occur if we get 4 and 2 or 3 and 3 or 2 and 4 with probability of occuring

$$= \tfrac{2}{21} \times \tfrac{1}{3} + \tfrac{2}{7} \times \tfrac{2}{7} + \tfrac{1}{3} \times \tfrac{2}{21}$$
$$= \tfrac{64}{441}$$

10 The three vectors are coplanar if the normal to **u** and **v** is parallel to the normal to **u** and **w** (or **v** and **w**).

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & 3\\ 1 & 2 & -3\end{vmatrix}$$
$$= \begin{vmatrix}2 & 3\\ 2 & -3\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 3\\ 1 & -3\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 2\\ 1 & 2\end{vmatrix}\mathbf{k}$$
$$= (-6-6)\mathbf{i} - (-6-3)\mathbf{j} + (4-2)\mathbf{k}$$
$$= -12\mathbf{i} + 9\mathbf{j} + 2\mathbf{k}$$

$$\text{and } \mathbf{u} \times \mathbf{w} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & 3\\ 1 & 2-\lambda & \lambda+1\end{vmatrix}$$
$$= \begin{vmatrix}2 & 3\\ 2-\lambda & \lambda+1\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 3\\ 1 & \lambda+1\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & 2\\ 1 & 2-\lambda\end{vmatrix}\mathbf{k}$$

$$= ((2\lambda+2)-(6-3\lambda))\,\mathbf{i}-((2\lambda+2)-3)\,\mathbf{j} + ((4-2\lambda)-2)\,\mathbf{k}$$
$$= (5\lambda-4)\mathbf{i}+(1-2\lambda)\mathbf{j}+(2-2\lambda)\mathbf{k}$$

The vectors are parallel if $\dfrac{5\lambda-4}{-12}=\dfrac{1-2\lambda}{9}=\dfrac{2-2\lambda}{2}$.

Using $\dfrac{1-2\lambda}{9}=\dfrac{2-2\lambda}{2}=1-\lambda$,
$$1-2\lambda = 9-9\lambda$$
$$\therefore\ \lambda=\tfrac{8}{7}$$
and $\lambda=\frac{8}{7}$ checks in the other equation.

11 **a** $u_1=285,\ \ u_{10}=213$

Now $u_{10}=u_1+9d\ \ \therefore\ 213=285+9d$
$$\therefore\ 9d=-72$$
$$\therefore\ d=-8$$

Thus, $u_n=285+(n-1)(-8)$
$$=285-8n+8$$
$$=293-8n$$

b $u_n>0$ for $293-8n>0$
$$\therefore\ 8n<293$$
$$\therefore\ n<36\tfrac{5}{8}$$
So, 36 terms are positive.

12
$$3\binom{n}{2}=\binom{n}{3}$$
$$\therefore\ 3\frac{n(n-1)}{2\times1}=\frac{n(n-1)(n-2)}{3\times2\times1}$$
$$\therefore\ 9n(n-1)=n(n-1)(n-2)$$
Since $n\geqslant3$, n and $n-1$ cannot be 0.

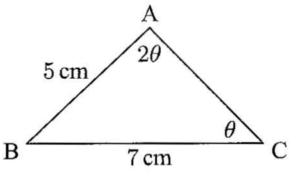
$\therefore\ 9=n-2$ and so $n=11$.

13 Since 12 customers per hour arrive, 6 will arrive each $\frac{1}{2}$ hour.

Let X = number of customers that arrive each $\frac{1}{2}$ hour.

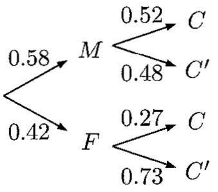
So, $X\sim\text{Po}(6)$ and $\text{P}(X=5)=\dfrac{6^5e^{-6}}{5!}\approx0.161$

14



By the sine rule, $\dfrac{\sin2\theta}{7}=\dfrac{\sin\theta}{5}$
$$\therefore\ 5\sin2\theta=7\sin\theta$$
$$\therefore\ 10\sin\theta\cos\theta-7\sin\theta=0$$
$$\therefore\ \sin\theta(10\cos\theta-7)=0$$
$$\therefore\ \cos\theta=\tfrac{7}{10}\qquad\{\text{as }\sin\theta\neq0\}$$
$$\therefore\ \theta=\arccos(0.7)\approx45.6^\circ$$

15 $C\equiv$ owns a car



a P(student does not have a car)
$$=0.58\times0.48+0.42\times0.73$$
$$=0.585$$

b $\text{P}(M\mid C')=\dfrac{\text{P}(M\cap C')}{\text{P}(C')}$
$$=\frac{0.58\times0.48}{0.585}\qquad\{\text{using }\mathbf{a}\}$$
$$\approx0.476$$

16 P_n is: "$\dfrac{d^n}{dx^n}(xe^x)=(x+n)e^x$", $n\in\mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n=1$, $\dfrac{d}{dx}(xe^x)=1e^x+xe^x$
$$=e^x(x+1)$$
$$=(x+1)e^x\quad\therefore\ P_1\text{ is true.}$$

(2) If P_k is true, $\dfrac{d^k}{dx^k}(xe^x)=(x+k)e^x$
$$\therefore\ \frac{d^{k+1}}{dx^{k+1}}(xe^x)=\frac{d}{dx}(x+k)e^x=1e^x+(x+k)e^x$$
$$=e^x(x+k+1)$$
$$=e^x(x+[k+1])$$

Thus P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore\ P_n$ is true for all $n\in\mathbb{Z}^+$

{Principle of mathematical induction}

17 **a**
$$\int_0^a x\sqrt{1-x^2}\,dx=0.2$$
$$\therefore\ -\tfrac{1}{2}\int_0^a(1-x^2)^{\frac{1}{2}}(-2x)\,dx=0.2$$
$$\therefore\ -\tfrac{1}{2}\left[\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a=0.2$$
$$\therefore\ -\tfrac{1}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^a=0.2$$
$$\therefore\ (1-a^2)^{\frac{3}{2}}-1=-0.6$$
$$\therefore\ (1-a^2)^{\frac{3}{2}}=0.4$$
$$\therefore\ 1-a^2=(0.4)^{\frac{2}{3}}$$
$$\therefore\ a^2=1-(0.4)^{\frac{2}{3}}$$
$$\therefore\ a\approx0.676,\ \text{ as }a>0$$

b
$$\int_0^a\frac{x}{x^2+1}\,dx=1$$
$$\therefore\ \tfrac{1}{2}\int_0^a\frac{2x}{x^2+1}\,dx=1$$
$$\therefore\ \int_0^a\frac{2x}{x^2+1}\,dx=2$$
$$\therefore\ \left[\ln\left|x^2+1\right|\right]_0^a=2$$
$$\therefore\ \ln\left|a^2+1\right|=2$$
$$\therefore\ a^2+1=e^2\qquad\{\text{since }a^2+1>0\}$$
$$\therefore\ a=\pm\sqrt{e^2-1}$$
$$\therefore\ a=\sqrt{e^2-1}\qquad\{\text{since }a>0\}$$

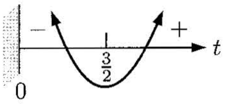
18 Area of the parallelogram is $|\mathbf{a}\times\mathbf{b}|$ where
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-4&1\\2&1&-5\end{vmatrix}$$
$$=(20-1)\mathbf{i}-(-15-2)\mathbf{j}+(3--8)\mathbf{k}$$
$$=19\mathbf{i}-17\mathbf{j}+11\mathbf{k}$$
Area of parallelogram is $\sqrt{19^2+17^2+11^2}\approx27.8$ units^2

19 **a** $v = 4t^3 - 9t^2 + 2, \quad t \geqslant 0$

$\therefore \; a = 12t^2 - 18t \text{ m s}^{-2}$

b $a = 6t(2t-3)$

with sign diagram:



v is a minimum where $\dfrac{dv}{dt} = 0$

$$\therefore \; a = 0, \quad t = \tfrac{3}{2}$$

$$\therefore \; v_{\min} = 4\left(\tfrac{27}{8}\right) - 9\left(\tfrac{9}{4}\right) + 2$$

$$= \tfrac{27}{2} - \tfrac{81}{4} + 2$$

$$= -\tfrac{19}{4} \text{ m s}^{-1}$$

c $s = \int v\,dt = \dfrac{4t^4}{4} - \dfrac{9t^3}{3} + 2t + c$

$= t^4 - 3t^3 + 2t + c$

But $s(0) = -6$ so $c = -6$

$\therefore \; s = t^4 - 3t^3 + 2t - 6$ m

d $s = 0$ when $t^3(t-3) + 2(t-3) = 0$

$$\therefore \; (t-3)(t^3+2) = 0$$

$$\therefore \; t = 3 \text{ or } -\sqrt[3]{2}$$

$$\therefore \; t = 3 \quad \{\text{as } t \geqslant 0\}$$

$\therefore$ first passes through O when $t = 3$ s.

e Total distance travelled in [0, 5]

$= \int_0^5 |v| \, dt$

$= \int_0^5 \left|4t^3 - 9t^2 + 2\right| \, dt$

≈ 2140 m {using technology}

20 Let X be the score in the Biology exam.

$\therefore \; X \sim \text{N}(56, 30.512^2)$

a $\text{P}(X \geqslant 72) \approx 0.30$

About 30% gained a score of "6" or better.

b $\text{P}(X \geqslant 40) \approx 0.70$

About 70% passed the exam.

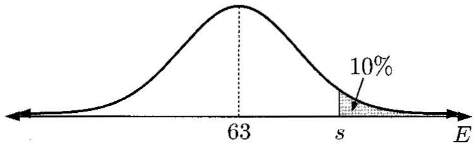
c $\text{P}(X \geqslant 94) \approx 0.1065$

So 10.65% of students would gain a mark of 94% or more, and Micah would just miss out on a score of 7, but he would get a score of 6.

d **i** Let E be the score in English then $E \sim \text{N}(63, 18.31^2)$ and $\text{P}(E \geqslant 87) \approx 0.0950$. So Micah scored better in English than in Biology.

ii Only 10% of the students receive a 7 in Biology. Let s be the mark required to get a 7 in English.

$\therefore \; \text{P}(E \geqslant s) = 0.1$ and so $\text{P}(E \leqslant s) = 0.9$



$$\therefore \; s \approx 86.5$$

A grade of 86.5% or better would get a 7 in English. Since Micah scored 87% in English, he would get a 7.

21 **a** $f(x) = x^n$

$\therefore \; f(x+h)$

$= (x+h)^n$

$= x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \ldots.$

$\ldots. + \binom{n}{n-2} x^2h^{n-2} + \binom{n}{n-1} xh^{n-1} + h^n$

b $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

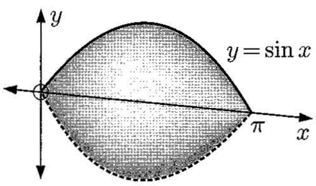
$$= \lim_{h \to 0} \frac{\left(\begin{matrix} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 \\ + \ldots. + nxh^{n-1} + h^n - x^n \end{matrix} \right)}{h}$$

$$= \lim_{h \to 0} \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2}h + \ldots. + h^{n-1}$$

$$= nx^{n-1} + 0 + 0 + \ldots. + 0$$

$$= nx^{n-1}$$

22



$$\text{Volume} = \pi \int_0^\pi y^2 \, dx$$

$$= \pi \int_0^\pi \sin^2 x \, dx$$

$$= \pi \int_0^\pi \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx$$

$$= \pi \left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^\pi$$

$$= \pi\left(\tfrac{\pi}{2} - 0 - 0 + 0\right)$$

$$= \tfrac{\pi^2}{2} \text{ units}^3$$

23 $f(x) = 2x\sin x, \;\; g(x) = x, \;\; 0 \leqslant x \leqslant 2\pi$

a They meet when $f(x) = g(x)$

$$\therefore \; 2x\sin x = x$$

$$\therefore \; x(2\sin x - 1) = 0$$

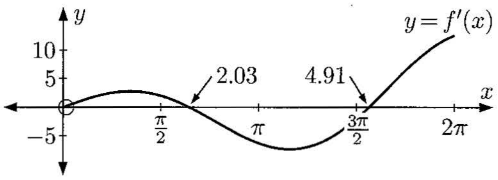
$$\therefore \; x = 0 \text{ or } \sin x = \tfrac{1}{2}$$

$$\therefore \; x = 0, \; \tfrac{\pi}{6}, \; \tfrac{5\pi}{6}$$

$\therefore$ they meet at (0, 0), $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{5\pi}{6}\right)$.

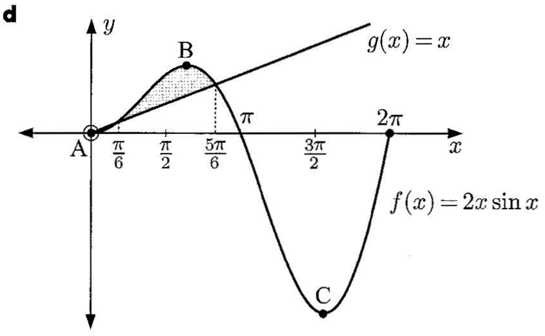
b $f'(x) = 2\sin x + 2x\cos x$ and so we have stationary points where $2\sin x + 2x\cos x = 0$

c Using technology we obtain the graph of $y = 2\sin x + 2x\cos x$:



The stationary points are at A(0, 0), B(2.03, 3.64), C(4.91, −9.63)

d



e We integrate by parts with $u = 2x \quad v' = \sin x$

$u' = 2 \quad v = -\cos x$

$$\therefore \; \int 2x \sin x \, dx = -2x\cos x - \int -2\cos x \, dx$$

$$= -2x\cos x + 2\sin x + c$$

f Total area

$$= \int_0^{\frac{\pi}{6}} (g(x) - f(x))\,dx + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (f(x) - g(x))\,dx$$

$$= \int_0^{\frac{\pi}{6}} (x - 2x\sin x)\,dx + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2x\sin x - x)\,dx$$

$$= \left[\frac{x^2}{2} + 2x\cos x - 2\sin x\right]_0^{\frac{\pi}{6}} + \left[-2x\cos x + 2\sin x - \frac{x^2}{2}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \tfrac{\pi^2}{72} + \tfrac{\pi}{3}\left(\tfrac{\sqrt{3}}{2}\right) - 1 - 0 + \left(-\tfrac{5\pi}{3}(-\tfrac{\sqrt{3}}{2}) + 1 - \tfrac{25\pi^2}{72}\right) - \left(-(\tfrac{\pi}{3})(\tfrac{\sqrt{3}}{2}) + 1 - \tfrac{\pi^2}{72}\right)$$

$$= \tfrac{\pi^2}{72} + \tfrac{\pi\sqrt{3}}{6} - 1 + \tfrac{5\pi\sqrt{3}}{6} - \tfrac{25\pi^2}{72} + \tfrac{\pi\sqrt{3}}{6} + \tfrac{\pi^2}{72}$$

$$= -\tfrac{23}{72}\pi^2 + \tfrac{7}{6}\pi\sqrt{3} - 1 \text{ units}^2$$

24 **a** Total number of students is 16.
We can select 7 from 16 in $\binom{16}{7} = 11\,440$ ways.

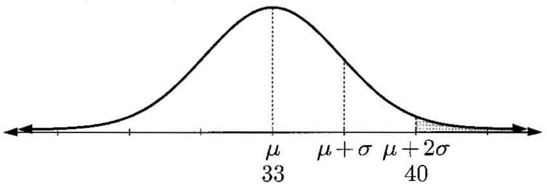
b The number of ways of selecting a committee with *both* Haakon and Josefine is $\binom{14}{5} = 2002$.
So, if they cannot both be on the committee the number of ways is $\binom{16}{7} - \binom{14}{5} = 9438$.

c If there are more boys than girls, the committee must be one of the following:

For 4 boys 3 girls, the number is $\binom{9}{4}\binom{7}{3}$.
For 5 boys 2 girls, the number is $\binom{9}{5}\binom{7}{2}$.
For 6 boys 1 girl, the number is $\binom{9}{6}\binom{7}{1}$.
For 7 boys 0 girls, the number is $\binom{9}{7}$.

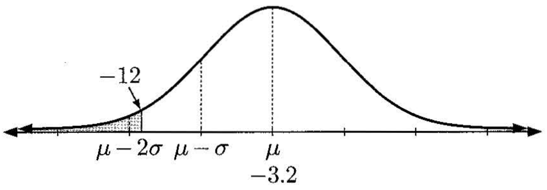
Total is $\binom{9}{4}\binom{7}{3} + \binom{9}{5}\binom{7}{2} + \binom{9}{6}\binom{7}{1} + \binom{9}{7} = 7680$.

25 **a** Let T_a be the maximum temperature in Adelaide so $T_a \sim \text{N}(33, 3.5^2)$.



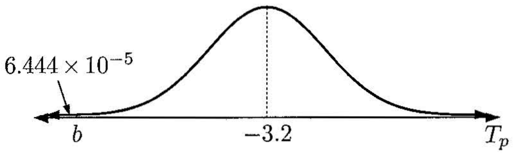
$\text{P}(T_a \geqslant 40) \approx 0.0228$
About 2.28% of January days in Adelaide will have temperatures above 40°C.

b Let T_p be the minimum temperature in Prague then $T_p \sim \text{N}(-3.2, 4.9^2)$ and $\text{P}(T_p < -12) \approx 0.0363$.



c Adelaide was 2 standard deviations above the mean while Prague was about 1.80 standard deviations below the mean. Hence the temperature in Adelaide was more extreme.

d $\text{P}(T_a \geqslant 46.4) \approx 6.444 \times 10^{-5}$
Let b be the corresponding extreme temperature in Prague.
$\therefore\ \text{P}(T_p \leqslant b) = 6.444 \times 10^{-5}$



$\therefore\ b \approx -22.0$
So, it would have to be −22.0°C in Prague.

SOLUTIONS TO TRIAL EXAMINATION 1

NO CALCULATOR

SECTION A

1 **a** $\sum \text{P}(X = x) = 1,$

$$\therefore\ \tfrac{1}{6} + 2k + \tfrac{1}{5}k + \tfrac{1}{3} + \tfrac{2}{5}k = 1$$
$$\therefore\ \tfrac{1}{2} + 2\tfrac{3}{5}k = 1$$
$$\therefore\ \tfrac{13}{5}k = \tfrac{1}{2}$$
$$\therefore\ k = \tfrac{5}{26}$$

b $\text{P}(0 < X < 4) = \text{P}(X = 1,\ 2, \text{ or } 3)$

$$= 2\left(\tfrac{5}{26}\right) + \tfrac{1}{5}\left(\tfrac{5}{26}\right) + \tfrac{1}{3}$$
$$= \tfrac{5}{13} + \tfrac{1}{26} + \tfrac{1}{3}$$
$$= \tfrac{11}{26} + \tfrac{1}{3}$$
$$= \tfrac{59}{78}$$

c $\text{E}(X + 1)$

$$= \text{E}(X) + \text{E}(1)$$
$$= \text{E}(X) + 1$$
$$= 0\left(\frac{1}{6}\right) + 1(2k) + 2\left(\frac{k}{5}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{2k}{5}\right) + 1$$
$$= 2k + \frac{2k}{5} + 1 + \frac{8k}{5} + 1$$
$$= 2 + 4k$$
$$= 2 + 4\left(\tfrac{5}{26}\right)$$
$$= 2 + \tfrac{10}{13}$$
$$= 2\tfrac{10}{13}$$

2 **a** $\alpha + \beta = -\tfrac{2}{5}$ and $\alpha\beta = -\tfrac{3}{5}$

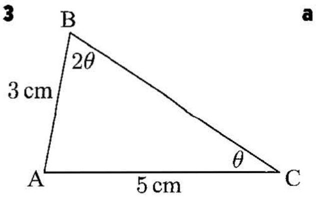
$$\therefore\ \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-\frac{2}{5}}{-\frac{3}{5}} = \tfrac{2}{3}$$

and $\dfrac{1}{\alpha\beta} = -\tfrac{5}{3}$

b All quadratic equations with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ have the form $a(x^2 - \tfrac{2}{3}x - \tfrac{5}{3}) = 0,\ \ a \neq 0$
which is equivalent to

$$a(3x^2 - 2x - 5) = 0, \quad a \neq 0.$$

3



a Using the sine rule,

$$\frac{\sin 2\theta}{5} = \frac{\sin\theta}{3}$$

$$\therefore \quad \frac{2\sin\theta\cos\theta}{5} = \frac{\sin\theta}{3}$$

$$\therefore \quad 6\sin\theta\cos\theta = 5\sin\theta$$

$$\therefore \quad \sin\theta(6\cos\theta - 5) = 0$$

$$\therefore \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = \tfrac{5}{6}$$

But $\sin\theta \neq 0$ as this would imply $\theta = 0$.

$\therefore \quad \cos\theta = \frac{5}{6}$.

b

$$\cos\theta = \tfrac{5}{6}$$

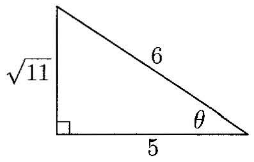
$$\therefore \quad \sin\theta = \tfrac{\sqrt{11}}{6}$$

$$\therefore \quad \sin \widehat{ABC} = \sin 2\theta$$

$$= 2\sin\theta\cos\theta$$

$$= 2(\tfrac{\sqrt{11}}{6})(\tfrac{5}{6})$$

$$= \tfrac{5\sqrt{11}}{18}$$



4

$$\left|\frac{3x-1}{x+2}\right| > 1, \quad x \neq -2$$

$$\therefore \quad \frac{|3x-1|}{|x+2|} > 1 \qquad \left\{\left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right\}$$

$$\therefore \quad |3x-1| > |x+2| \qquad \{\text{as } |x+2| \text{ is not negative}\}$$

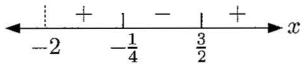
$$\therefore \quad |3x-1|^2 > |x+2|^2$$

$$\therefore \quad (3x-1)^2 > (x+2)^2 \qquad \{|a|^2 = a^2\}$$

$$\therefore \quad (3x-1)^2 - (x+2)^2 > 0$$

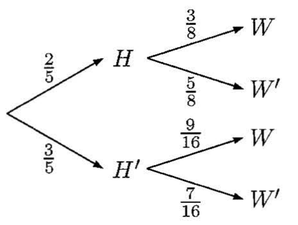
$$\therefore \quad (3x-1+x+2)(3x-1-x-2) > 0$$

$$\therefore \quad (4x+1)(2x-3) > 0$$



$$\therefore \quad x < -\tfrac{1}{4} \quad \text{or} \quad x > \tfrac{3}{2}, \quad x \neq -2$$

5 **a**



H = weather is hot
W = local team wins

b

$$P(H \mid W) = \frac{P(W \mid H)\,P(H)}{P(W)} \qquad \{\text{Bayes theorem}\}$$

$$= \left(\frac{\frac{3}{8}\times\frac{2}{5}}{\frac{3}{8}\times\frac{2}{5}+\frac{9}{16}\times\frac{3}{5}}\right)\frac{80}{80}$$

$$= \frac{12}{12+27}$$

$$= \tfrac{12}{39}$$

$$= \tfrac{4}{13}$$

6 **a** The graph touches the x-axis at $-\frac{1}{2}$.

$\therefore \quad (2x+1)^2$ is a factor of $f(x)$.

The graph also cuts the x-axis at 1.

$\therefore \quad (x-1)$ is a factor also.

Thus $f(x) = k(2x+1)^2(x-1), \ k \neq 0$

But $f(0) = -2$, so $k(1)^2(-1) = -2$

$$\therefore \quad -k = -2$$

$$\therefore \quad k = 2$$

Thus

$$f(x) = 2(2x+1)^2(x-1)$$

$$= 2(4x^2+4x+1)(x-1)$$

$$= 2(4x^3-3x-1)$$

$$= 8x^3-6x-2$$

$$\therefore \quad a = 8, \ b = 0, \ c = -6, \ d = -2$$

b

$$g(x) = 2f(x-1)$$

$$\therefore \quad g(0) = 2f(-1)$$

$$= 2[2(-1)^2(-2)]$$

$$= 2 \times -4$$

$$= -8 \qquad \therefore \ y\text{-intercept is } -8.$$

c

$$g(x) = 2f(x-1) = 2[8(x-1)^3 - 6(x-1) - 2]$$

$$= 16(x-1)^3 - 12(x-1) - 4$$

$$\therefore \quad g'(x) = 48(x-1)^2 - 12$$

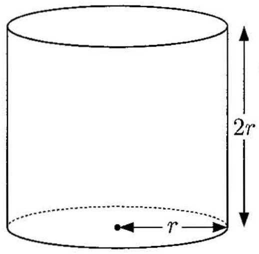
$$\therefore \quad g''(x) = 96(x-1)$$

$$\therefore \quad g''(x) = 0 \quad \text{when} \quad x = 1$$

Now $g(1) = 2f(0) = -4$

$\therefore \quad g(x)$ has point of inflection $(1, -4)$.

7



$$V = \pi r^2 h$$

$$= \pi r^2(2r)$$

$$= 2\pi r^3$$

$$\therefore \quad \frac{dV}{dt} = 6\pi r^2\,\frac{dr}{dt}$$

But when $r = 5$, $\dfrac{dV}{dt} = 5\pi$

$$\therefore \quad 5\pi = 6\pi \times 5^2\,\frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = \tfrac{1}{30}\ \text{cm s}^{-1}$$

at this instant.

Now surface area,

$$A = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + 2\pi r(2r)$$

$$= 6\pi r^2$$

$$\therefore \quad \frac{dA}{dt} = 12\pi r\,\frac{dr}{dt}$$

At the instant when $r = 5$,

$$\frac{dA}{dt} = 12\pi \times 5 \times \tfrac{1}{30} = 2\pi\ \text{cm}^2\,\text{s}^{-1}$$

$\therefore$ the area is increasing at $2\pi\ \text{cm}^2\,\text{s}^{-1}$.

8 Let $x = \sec\theta$

$$\therefore \quad \frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$\therefore \quad \int \frac{\sqrt{x^2-1}}{2x}\,dx$$

$$= \int \frac{\sqrt{\sec^2\theta - 1}}{2\sec\theta}\,\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan\theta}{2\sec\theta}\,\sec\theta\tan\theta\,d\theta$$

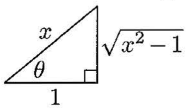
$$= \tfrac{1}{2}\int \tan^2\theta\,d\theta$$

$$= \tfrac{1}{2}\int (\sec^2\theta - 1)\,d\theta$$

$$= \tfrac{1}{2}(\tan\theta - \theta) + c$$

$$= \tfrac{1}{2}\sqrt{x^2-1} - \tfrac{1}{2}\arctan\left(\sqrt{x^2-1}\right) + c$$

$$\sec\theta = \frac{x}{1}$$



9 P_n is "if $y = xe^{-x}$ then $\dfrac{d^ny}{dx^n} = \dfrac{(-1)^{n+1}(n-x)}{e^x}$", for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$,
$$\begin{aligned}\frac{dy}{dx} &= 1e^{-x} + x(-e^{-x})\\ &= (1-x)e^{-x}\\ &= \frac{1-x}{e^x}\\ &= \frac{(-1)^{1+1}(1-x)}{e^x}\end{aligned}$$

$\therefore$ P_1 is true.

(2) If P_k is true, $\dfrac{d^ky}{dx^k} = \dfrac{(-1)^{k+1}(k-x)}{e^x}$

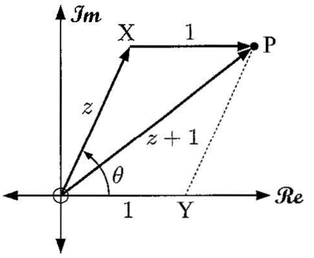
$$\begin{aligned}\therefore\ \frac{d^{k+1}y}{dx^{k+1}} &= (-1)^{k+1}\left[\frac{-1e^x-(k-x)e^x}{(e^x)^2}\right]\\ &= (-1)^{k+1}\left[\frac{e^x(-1-k+x)}{e^{2x}}\right]\\ &= (-1)^{k+1}\left[\frac{-(k+1-x)}{e^x}\right]\\ &= \frac{(-1)^{k+2}([k+1]-x)}{e^x}\end{aligned}$$

Thus P_1 is true, and P_{k+1} is true whenever P_k is true.

$\therefore$ P_n is true {Principle of mathematical induction}

SECTION B

10 **a**



$|z| = r$

b Since $\overrightarrow{XP}$ is parallel to the $\mathcal{R}e$ axis, $\widehat{OXP} = 180° - \theta$

In $\triangle$OXP,
$$\begin{aligned}\text{OP}^2 &= r^2 + 1^2 - 2r \times 1 \times \cos(180° - \theta)\\ &= r^2 + 1 - 2r\cos(180° - \theta)\\ &= r^2 + 1 + 2r\cos\theta\end{aligned}$$
$$\therefore\ \text{OP} = \sqrt{r^2 + 2r\cos\theta + 1}$$

c Suppose $\arg(z+1) = \alpha$. Locate Y so that $\overrightarrow{YP} = z$ and OXPY is a parallelogram.

In $\triangle$OYP,
$$\begin{aligned}\cos\alpha &= \frac{1^2 + \text{OP}^2 - r^2}{2(1)(\text{OP})} \quad \text{\{cosine rule\}}\\ &= \frac{1 + \cancel{r^2} + 2r\cos\theta + 1 - \cancel{r^2}}{2\sqrt{r^2 + 2r\cos\theta + 1}}\\ &= \frac{2 + 2r\cos\theta}{2\sqrt{r^2 + 2r\cos\theta + 1}}\\ &= \frac{1 + r\cos\theta}{\sqrt{r^2 + 2r\cos\theta + 1}}\end{aligned}$$

d If $|z| = 1$ then $r = 1$
$$\begin{aligned}\therefore\ \cos\alpha &= \frac{1 + \cos\theta}{\sqrt{1 + 2\cos\theta + 1}}\\ &= \frac{1 + \cos\theta}{\sqrt{2 + 2\cos\theta}}\\ &= \frac{1 + \cos\theta}{\sqrt{2}\sqrt{1 + \cos\theta}}\\ &= \sqrt{\frac{1 + \cos\theta}{2}}\\ &= \sqrt{\frac{1 + (2\cos^2\left(\frac{\theta}{2}\right) - 1)}{2}}\\ &= \sqrt{\frac{\cancel{2}\cos^2\left(\frac{\theta}{2}\right)}{\cancel{2}}}\\ &= \cos\left(\tfrac{\theta}{2}\right) \quad \text{\{since } \tfrac{\theta}{2} \text{ is acute\}}\end{aligned}$$

Thus $\alpha = \frac{\theta}{2}$

e When $r = 1$, OXPY is a rhombus and (OP) bisects $\widehat{XOY}$

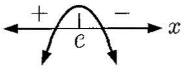
$\therefore$ $\alpha = \frac{\theta}{2}$.

11 $f(x) = \dfrac{k\ln x}{x}$, $k > 0$, $x > 0$.

a
$$\begin{aligned}f'(x) &= \frac{k\left(\frac{1}{x}\right)x - k\ln x(1)}{x^2}\\ &= \frac{k(1 - \ln x)}{x^2}\end{aligned}$$

$\therefore$ $f'(x) = 0$ when $\ln x = 1$,
which is when $x = e$.

Sign diagram for $f'(x)$:



Now $f(e) = \dfrac{k}{e}$

$\therefore$ there is a local maximum at $\left(e, \dfrac{k}{e}\right)$.

b
$$\begin{aligned}f''(x) &= k\left[\frac{-\frac{1}{x}(x^2) - (1 - \ln x)2x}{x^4}\right]\\ &= \frac{k[-x - 2x + 2x\ln x]}{x^4}\\ &= \frac{k(2\ln x - 3)}{x^3}\end{aligned}$$

$\therefore$ $f''(x) = 0$ when $\ln x = \frac{3}{2}$
which is when $x = e^{\frac{3}{2}}$

$\therefore$ the point of inflection is at $\left(e^{\frac{3}{2}}, \dfrac{3k}{2e^{\frac{3}{2}}}\right)$.

c If $\displaystyle\int_1^{e^2} \frac{k\ln x}{x}\,dx = 10$ then

$$k\int_1^{e^2} [\ln x]^1\left(\frac{1}{x}\right)dx = 10$$
$$\therefore\ k\left[\frac{[\ln x]^2}{2}\right]_1^{e^2} = 10$$
$$\therefore\ \frac{[\ln e^2]^2}{2} - \cancel{\frac{[\ln 1]^2}{2}} = \frac{10}{k}$$
$$\therefore\ 2 = \frac{10}{k}$$
$$\therefore\ k = 5$$

d We integrate by parts with $u = (\ln x)^2$ $\quad v' = \dfrac{1}{x^2}$

$u' = 2(\ln x)^1\dfrac{1}{x}$ $\quad v = -\dfrac{1}{x}$

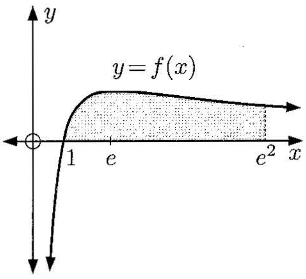
$$\begin{aligned}\therefore\ \int \frac{(\ln x)^2}{x^2}\,dx &= uv - \int u'v\,dx\\ &= -\frac{1}{x}(\ln x)^2 - \int \frac{2\ln x}{x} \times -\frac{1}{x}\,dx\\ &= -\frac{1}{x}(\ln x)^2 + 2\int \frac{\ln x}{x^2}\,dx\end{aligned}$$

We again use integration by parts, this time with

$$u = \ln x \qquad v' = \frac{1}{x^2}$$
$$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$$

$$\therefore \int \frac{(\ln x)^2}{x^2}\,dx$$
$$= -\frac{1}{x}(\ln x)^2 + 2\left[-\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx\right]$$
$$= -\frac{1}{x}(\ln x)^2 - \frac{2\ln x}{x} + 2\int \frac{1}{x^2}\,dx$$
$$= -\frac{1}{x}(\ln x)^2 - \frac{2\ln x}{x} - \frac{2}{x} + c$$

e



When $k = 1$,

$f(x) = \dfrac{\ln x}{x}$,

the local maximum is at $\left(e, \dfrac{1}{e}\right)$,

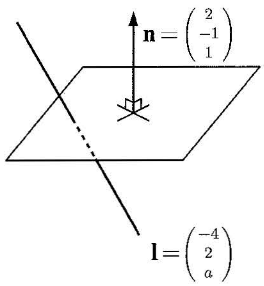
and $f(e^2) = \dfrac{2}{e^2}$.

The x-intercept is when $\ln x = 0$, which is when $x = 1$

$\therefore$ the volume of the solid of revolution

$$V = \pi \int_1^{e^2} y^2\,dx$$
$$= \pi \int_1^{e^2} \frac{(\ln x)^2}{x^2}\,dx$$
$$= \pi \left[-\frac{1}{x}(\ln x)^2 - \frac{2\ln x}{x} - \frac{2}{x}\right]_1^{e^2}$$
$$= \pi\left[\left(-\frac{4}{e^2} - \frac{4}{e^2} - \frac{2}{e^2}\right) - (0+0-2)\right]$$
$$= \pi\left(2 - \frac{10}{e^2}\right) \text{ units}^3$$

12 **a**



i Since the line is perpendicular to the plane, **n** is parallel to **l**

$$\therefore \begin{pmatrix}-4\\2\\a\end{pmatrix} = k\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$\therefore$ $2k = -4$, $-k = 2$, and $k = a$

Thus $a = k = -2$

ii The line meets the plane when

$$2(1-4\lambda) - (2+2\lambda) + (3-2\lambda) = 15$$
$$\therefore 2 - 8\lambda - 2 - 2\lambda + 3 - 2\lambda = 15$$
$$\therefore -12\lambda = 12$$
$$\therefore \lambda = -1$$

and $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-4\\2\\-2\end{pmatrix} = \begin{pmatrix}5\\0\\5\end{pmatrix}$

$\therefore$ they meet at $(5, 0, 5)$.

b **i** $\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OP}$

$$= \begin{pmatrix}1-4\lambda\\2+2\lambda\\3-2\lambda\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix}$$
$$= \begin{pmatrix}-2-4\lambda\\3+2\lambda\\1-2\lambda\end{pmatrix}$$

ii $\overrightarrow{PA} \perp \mathbf{l}$ so $\overrightarrow{PA} \bullet \mathbf{l} = 0$

$$\therefore (-2-4\lambda)(-4) + (3+2\lambda)(2) + (1-2\lambda)(-2) = 0$$
$$\therefore 8 + 16\lambda + 6 + 4\lambda - 2 + 4\lambda = 0$$
$$\therefore 24\lambda = -12$$
$$\therefore \lambda = -\tfrac{1}{2}$$

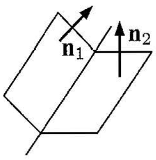
iii When $\lambda = -\frac{1}{2}$, $\overrightarrow{OA} = \begin{pmatrix}3\\1\\4\end{pmatrix}$

$\therefore$ the foot of the normal is $(3, 1, 4)$.

iv $\overrightarrow{PA} = \begin{pmatrix}0\\2\\2\end{pmatrix}$ and $|\overrightarrow{PA}| = \sqrt{0+4+4}$
$= \sqrt{8} = 2\sqrt{2}$

$\therefore$ the shortest distance is $2\sqrt{2}$ units.

c The third plane has direction vector



$$= \mathbf{n}_1 \times \mathbf{n}_2$$
$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 1\\ 1 & 3 & -2\end{vmatrix}$$
$$= \begin{vmatrix}-1 & 1\\ 3 & -2\end{vmatrix}\mathbf{i} - \begin{vmatrix}2 & 1\\ 1 & -2\end{vmatrix}\mathbf{j} + \begin{vmatrix}2 & -1\\ 1 & 3\end{vmatrix}\mathbf{k}$$
$$= (2-3)\mathbf{i} - (-4-1)\mathbf{j} + (6+1)\mathbf{k}$$
$$= -\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

$\therefore$ since $(-5, 0, 0)$ lies on the plane, its equation is $-x + 5y + 7z = -(-5) + 5(0) + 7(0)$ which is $x - 5y - 7z = -5$.

13 **a**

$$f(x) = \frac{ae^{-x}}{b - ae^{-x}}$$
$$\therefore f'(x) = \frac{(-ae^{-x})(b - ae^{-x}) - ae^{-x}(ae^{-x})}{(b - ae^{-x})^2}$$
$$= \frac{-abe^{-x} \cancel{+a^2e^{-2x}} \cancel{-a^2e^{-2x}}}{(b - ae^{-x})^2}$$
$$= \frac{-abe^{-x}}{(b - ae^{-x})^2}$$

b $f''(x)$

$$= \frac{(abe^{-x})(b - ae^{-x})^2 - (-abe^{-x})2(b - ae^{-x})^1(ae^{-x})}{(b - ae^{-x})^4}$$
$$= \frac{(abe^{-x})(b - ae^{-x}) + 2a^2be^{-2x}}{(b - ae^{-x})^3}$$
$$= \frac{ab^2e^{-x} - a^2be^{-2x} + 2a^2be^{-2x}}{(b - ae^{-x})^3}$$
$$= \frac{ab^2e^{-x} + a^2be^{-2x}}{(b - ae^{-x})^3}$$
$$= \frac{abe^{-x}(b + ae^{-x})}{(b - ae^{-x})^3}$$

As $a > 0$, $b > 0$, $e^{-x} > 0$, and $b + ae^{-x} > 0$, $f''(x)$ cannot be 0 for any $x \in \mathbb{R}$.

c $f(x)$ is undefined when $b - ae^{-x} = 0$

$$\therefore ae^{-x} = b$$
$$\therefore e^x = \frac{a}{b}$$
$$\therefore x = \ln\left(\frac{a}{b}\right)$$

$\therefore$ $x = \ln\left(\dfrac{a}{b}\right)$ is a vertical asymptote.

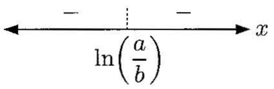
Also $f(x) = \left(\dfrac{ae^{-x}}{b - ae^{-x}}\right)\dfrac{e^x}{e^x} = \dfrac{a}{be^x - a}$

As $x \to \infty$, $f(x) \to 0$ {since $a, b > 0$}

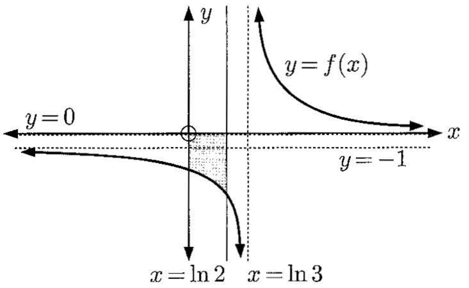
As $x \to -\infty$, $f(x) \to -1$

$\therefore$ $y = 0$ and $y = -1$ are horizontal asymptotes.

d Sign diagram for $f'(x)$:



e **i** If $a = 3$ and $b = 1$, the vertical asymptote is $x = \ln 3$.



ii The shaded area $= -\displaystyle\int_0^{\ln 2} \frac{3e^{-x}}{1-3e^{-x}}\,dx$

If we let $u = 1 - 3e^{-x}$

then $\dfrac{du}{dx} = 3e^{-x}$,

$u(0) = -2$, and

$u(\ln 2) = -\frac{1}{2}$

$$\therefore \text{ the area} = -\int_0^{\ln 2} \frac{1}{u}\frac{du}{dx}\,dx$$
$$= -\int_{-2}^{-\frac{1}{2}} \frac{1}{u}\,du$$
$$= -\Big[\ln|u|\Big]_{-2}^{-\frac{1}{2}}$$
$$= -(\ln\left(\tfrac{1}{2}\right) - \ln 2)$$
$$= \ln 2 + \ln 2$$
$$= \ln 4 \text{ units}^2$$

CALCULATOR

SECTION A

1 **a** $\displaystyle\prod_{n=1}^{10}(nx-1) = (x-1)(2x-1)(3x-1)\,....\,(10x-1)$

$$= 1(x-1)2(x-\tfrac{1}{2})3(x-\tfrac{1}{3})\,....\,10(x-\tfrac{1}{10})$$
$$= 10!(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3})\,....\,(x-\tfrac{1}{10})$$
$$\therefore \quad k = 10! = 3\,628\,800$$

b Using the 'sum and product of roots theorem'

$$(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3})\,....\,(x-\tfrac{1}{10})$$
$$= x^{10} - (1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{10})x^9 + ...$$

$\therefore$ the coefficient of x^9 in $\displaystyle\prod_{n=1}^{10}(nx-1)$ is

$$-10!(1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{10}) = -10\,628\,640$$

2 **a** Area of shaded segment

= area of sector − area of triangle

$= \frac{1}{2}r^2\theta - \frac{1}{2}rr\sin\theta$ $\quad\{\frac{1}{2}ab\sin c$ formula$\}$

$= \frac{1}{2}r^2(\theta - \sin\theta)$, θ in radians

b shaded area : unshaded area

$= \frac{1}{2}r^2(\theta - \sin\theta) : \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(\theta - \sin\theta) : \pi - \frac{1}{2}(\theta - \sin\theta)$

The ratio is $1 : 3$ when

$\pi - \frac{1}{2}(\theta - \sin\theta) = 3\left(\frac{1}{2}\right)(\theta - \sin\theta)$

$\therefore \quad 2(\theta - \sin\theta) = \pi$

$\therefore \quad \theta - \sin\theta - \frac{\pi}{2} = 0$

We use technology to graph $f(\theta) = \theta - \sin\theta - \frac{\pi}{2}$ where it is clear that θ is obtuse.

We hence find $\theta \approx 2.309\,881\,5$ radians

$\therefore \quad \theta \approx 132.3°$

3 **a** For $f(x)$ to be well defined,

$$\int_0^2 k(x^2+3)\,dx = 1$$
$$\therefore \quad k\left[\frac{x^3}{3} + 3x\right]_0^2 = 1$$
$$\therefore \quad k\left(\tfrac{8}{3} + 6 - 0\right) = 1$$
$$\therefore \quad k\left(\tfrac{26}{3}\right) = 1$$
$$\therefore \quad k = \tfrac{3}{26}$$

b $\displaystyle P(X \geqslant 0.6) = \int_{0.6}^{2} \tfrac{3}{26}(x^2+3)\,dx$

$= 0.784$ $\quad$ {using technology}

c The median m is such that

$$\int_0^m \tfrac{3}{26}(x^2+3)\,dx = \tfrac{1}{2}$$
$$\therefore \quad \left[\frac{x^3}{3} + 3x\right]_0^m = \tfrac{1}{2} \times \tfrac{26}{3}$$
$$\therefore \quad \frac{m^3}{3} + 3m = \tfrac{13}{3}$$
$$\therefore \quad m^3 + 9m - 13 = 0$$

$\therefore \quad m \approx 1.235$ $\quad$ {using technology}

4 $X \sim \mathrm{B}(12, p)$

$\therefore \quad \mathrm{P}(X = r) = \binom{12}{r}p^r(1-p)^{12-r}$ for $r = 0, 1, 2, ..., 12$

$\therefore \quad \mathrm{P}(X = 3) = \binom{12}{3}p^3(1-p)^9 = 0.1$

$\therefore \quad 220p^3(1-p)^9 = 0.1$

$\therefore \quad p^3(1-p)^9 = \frac{1}{2200}$

We use technology to solve this for $0 \leqslant p \leqslant 1$.

$\Rightarrow \quad p \approx 0.109$ or 0.442

5 **a** $2\cos 2\theta + 1 = 5\sin\theta, \quad \frac{\pi}{2} < \theta < \pi$

$\therefore \quad 2(1 - 2\sin^2\theta) + 1 - 5\sin\theta = 0$

$\therefore \quad 2 - 4\sin^2\theta + 1 - 5\sin\theta = 0$

$\therefore \quad 4\sin^2\theta + 5\sin\theta - 3 = 0$

b $\therefore \quad \sin\theta = \dfrac{-5 \pm \sqrt{25 - 4(4)(-3)}}{8}$

$\therefore \quad \sin\theta = \dfrac{-5 \pm \sqrt{73}}{8}$

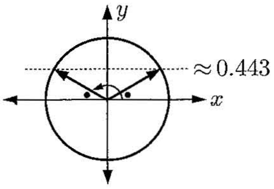
$\therefore \quad \sin\theta = \dfrac{\sqrt{73} - 5}{8}$ $\quad\left\{\text{as } \dfrac{-5 - \sqrt{73}}{8} < -1\right\}$

$\approx 0.443\,000$

As $\frac{\pi}{2} < \theta < \pi$,

$\theta \approx \pi - \arcsin(0.443)$

$\therefore \quad \theta \approx 2.683^c$



6 $X \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(X \geqslant 20) \approx 0.386$$
$$\therefore\ \mathrm{P}\left(\frac{X-\mu}{\sigma} \geqslant \frac{20-\mu}{\sigma}\right) \approx 0.386$$
$$\therefore\ \mathrm{P}\left(Z \geqslant \frac{20-\mu}{\sigma}\right) \approx 0.386$$
$$\therefore\ \mathrm{P}\left(Z \leqslant \frac{20-\mu}{\sigma}\right) \approx 0.614$$
$$\therefore\ \frac{20-\mu}{\sigma} \approx 0.289\,76$$
$$\therefore\ 20-\mu \approx 0.289\,76\sigma \quad .... \text{ (1)}$$

Also $\mathrm{P}(X \geqslant 25) \approx 0.183$

$$\therefore\ \mathrm{P}\left(Z \geqslant \frac{25-\mu}{\sigma}\right) \approx 0.183$$
$$\therefore\ \mathrm{P}\left(Z \leqslant \frac{25-\mu}{\sigma}\right) \approx 0.817$$
$$\therefore\ \frac{25-\mu}{\sigma} \approx 0.903\,99$$
$$\therefore\ 25-\mu \approx 0.903\,99\sigma \quad .... \text{ (2)}$$

Solving (1) and (2) simultaneously,

$$\begin{array}{r} 25-\mu \approx 0.903\,99\sigma \\ -20+\mu \approx -0.289\,76\sigma \\ \hline 5 \approx 0.614\,23\sigma \end{array}$$

$\therefore\ \sigma \approx 8.1403$

and $\mu \approx 20 - 0.289\,76 \times 8.1403$

≈ 17.641

So, $\mu \approx 17.6$ and $\sigma \approx 8.14$

7 $\log_a 3 = 7$

a $\log_a 27 = \log_a 3^3$
$= 3\log_a 3$
$= 3 \times 7$
$= 21$

b Let $\log_{\sqrt{a}} 3 = x$
$\therefore\ 3 = (\sqrt{a})^x$
$\therefore\ 3 = (a^{\frac{1}{2}})^x$
$\therefore\ a^{\frac{x}{2}} = 3$
$\therefore\ \frac{x}{2} = \log_a 3$
$\therefore\ \frac{x}{2} = 7$
$\therefore\ x = 14$

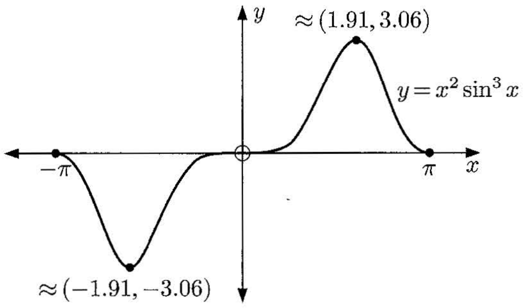
c Since $\log_a 3 = 7$,
$3 = a^7$
$\therefore\ a = 3^{\frac{1}{7}}$
$\therefore\ a \approx 1.17$

8 **a** $f(x) = x^2 \sin^3 x$

$\therefore\ f(-x) = (-x)^2[\sin(-x)]^3$
$= x^2 \times [-\sin x]^3$
$= x^2 \times -\sin^3 x$
$= -x^2 \sin^3 x$
$= -f(x)$ for all x

$\therefore\ f(x)$ is an odd function.

b



c $\int_{-a}^{a} f(x)\,dx = 0$ since $f(x)$ is odd.

This is because $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_{-a}^0 f(x)\,dx$

In the last integral, let $u = -x$, so $du = -dx$

$$\therefore\ \int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^0 f(-u)(-du)$$
$$= \int_0^a f(x)\,dx + \int_0^a f(-u)\,du$$
$$= \int_0^a f(x)\,dx - \int_0^a f(u)\,du$$

{since $f(x)$ is odd}

$$= 0$$

9 **a** Using technology:

i $\mu \approx 72.67$

ii $\sigma^2 \approx 3.559^2$
≈ 12.67

b Let X = the weight of a scallop

$\therefore\ X \sim \mathrm{N}(72.67, 12.67)$

$\therefore\ \mathrm{P}(70 < X < 80) \approx 0.754$

c The answer to **b** is unreliable, as the sample in **a** is very small.

SECTION B

10 **a** Let $u = \sqrt{x-1}$

so $u^2 = x - 1$ and $2u\,du = dx$

$$\therefore\ \int \frac{\sqrt{x-1}}{x}\,dx$$
$$= \int \frac{u}{u^2+1}\,2u\,du$$
$$= 2\int \frac{u^2}{u^2+1}\,du$$
$$= 2\int \frac{u^2+1-1}{u^2+1}\,du$$
$$= 2\int \left(1 - \frac{1}{u^2+1}\right) du$$
$$= 2(u - \arctan u) + c$$
$$= 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + c$$

b $f(x)$ is defined for $x - 1 \geqslant 0$ and $x \neq 0$

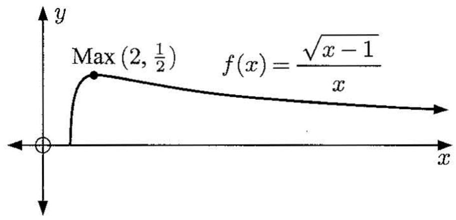
$\therefore$ the domain of f is $x \geqslant 1$.

$\therefore\ \int_{-1}^{1} \frac{\sqrt{x-1}}{x}\,dx$ does not exist.

c $\int_1^2 \frac{\sqrt{x-1}}{x}\,dx$

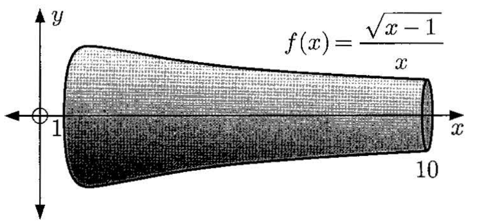
$$= \left[2\sqrt{x-1} - 2\arctan\sqrt{x-1}\,\right]_1^2$$
$$= 2(1) - 2\arctan(1) - 2(0) + 2\arctan(0)$$
$$= 2 - 2\left(\tfrac{\pi}{4}\right) - 0 + 0$$
$$= 2 - \tfrac{\pi}{2}$$

d **i**



ii The local maximum is at $(2, \frac{1}{2})$.
{using technology}

e **i**



ii
$$\text{Volume} = \pi\int_1^{10} y^2\,dx$$
$$= \pi\int_1^{10}\frac{x-1}{x^2}\,dx$$
$$= \pi\int_1^{10}\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$
$$= \pi\left[\ln|x|+\frac{1}{x}\right]_1^{10}$$
$$= \pi(\ln 10+\tfrac{1}{10}-\ln 1-1)$$
$$= \pi(\ln 10-\tfrac{9}{10})$$

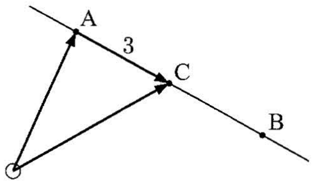
11 **a** $\mathbf{a}=\overrightarrow{OA}=\begin{pmatrix}-4\\12\\8\end{pmatrix}$ $\quad\mathbf{b}=\overrightarrow{OB}=\begin{pmatrix}4\\8\\0\end{pmatrix}$

$$\overrightarrow{OM}=\tfrac{1}{4}\mathbf{a}+\tfrac{3}{4}\mathbf{b}$$
$$=\tfrac{1}{4}\begin{pmatrix}-4\\12\\8\end{pmatrix}+\tfrac{3}{4}\begin{pmatrix}4\\8\\0\end{pmatrix}$$
$$=\begin{pmatrix}-1\\3\\2\end{pmatrix}+\begin{pmatrix}3\\6\\0\end{pmatrix}=\begin{pmatrix}2\\9\\2\end{pmatrix}$$

$\therefore$ M is $(2,\,9,\,2)$.

b $$\overrightarrow{AM}:\overrightarrow{MB}=\begin{pmatrix}6\\-3\\-6\end{pmatrix}:\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$
$$=3\begin{pmatrix}2\\-1\\-2\end{pmatrix}:\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$
$$=3:1$$

c



Let the point be C.

Now $\overrightarrow{AB}=\begin{pmatrix}8\\-4\\-8\end{pmatrix}$

$\therefore\ |\,\overrightarrow{AB}\,|=\sqrt{64+16+64}=12$

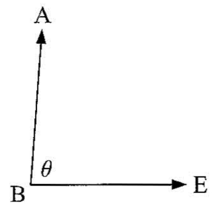
$$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$$
$$=\begin{pmatrix}-4\\12\\8\end{pmatrix}+3\times\tfrac{1}{12}\times\begin{pmatrix}8\\-4\\-8\end{pmatrix}$$
$$=\begin{pmatrix}-4\\12\\8\end{pmatrix}+\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$
$$=\begin{pmatrix}-2\\11\\6\end{pmatrix}$$

$\therefore$ the point is $(-2,\,11,\,6)$.

d A normal to the plane is
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-4&12&8\\4&8&0\end{vmatrix}$$
$$=\begin{vmatrix}12&8\\8&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}-4&8\\4&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}-4&12\\4&8\end{vmatrix}\mathbf{k}$$
$$=(0-64)\mathbf{i}-(0-32)\mathbf{j}+(-32-48)\mathbf{k}$$
$$=-64\mathbf{i}+32\mathbf{j}-80\mathbf{k}$$
$$=-16(4\mathbf{i}-2\mathbf{j}+5\mathbf{k})$$

Since the plane passes through O, it has equation $4x-2y+5z=0$.

e



$$\overrightarrow{BA}=\begin{pmatrix}-8\\4\\8\end{pmatrix},\quad\overrightarrow{BE}=\begin{pmatrix}-4\\-8\\1\end{pmatrix}$$
$$\cos\theta=\frac{\overrightarrow{BA}\bullet\overrightarrow{BE}}{|\,\overrightarrow{BA}\,|\,|\,\overrightarrow{BE}\,|}$$
$$=\frac{32-32+8}{12\times\sqrt{16+64+1}}$$
$$=\frac{8}{12\times 9}$$
$$=\tfrac{2}{27}$$
$$\therefore\ \theta=\arccos\left(\tfrac{2}{27}\right)$$
$$\therefore\ \widehat{ABE}\approx 85.8^\circ$$

12 **a** $z=r\operatorname{cis}\theta=re^{i\theta}$

$\therefore\ z^n=(re^{i\theta})^n$

$=r^ne^{i\theta n}$

$=r^n\operatorname{cis}(n\theta)$, which is De Moivre's theorem.

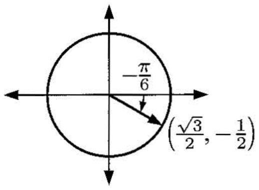
b $z^{-2}=r^{-2}\operatorname{cis}(-2\theta)$

$\therefore\ \left|z^{-2}\right|=r^{-2}=\dfrac{1}{r^2}$ and $\arg(z^{-2})=-2\theta$

c **i** $z=\sqrt{3}-i$ has $|z|=\sqrt{3+1}=2$

$\therefore\ z=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$

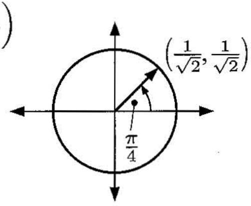
$\therefore\ z=2\operatorname{cis}\left(-\frac{\pi}{6}\right)$



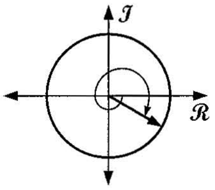
$w=2+2i$ has $|w|=\sqrt{4+4}=2\sqrt{2}$

$\therefore\ w=2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

$\therefore\ w=2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$



ii
$$\frac{z^7}{w^4}=\frac{\left[2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right]^7}{\left[2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^4}$$
$$=\frac{2^7\operatorname{cis}\left(-\frac{7\pi}{6}\right)}{2^6\operatorname{cis}(\pi)}$$
$$=2\operatorname{cis}\left(-\tfrac{13\pi}{6}\right)$$
$$=2\operatorname{cis}\left(-\tfrac{\pi}{6}\right)$$
$$=z$$



d $\cos 3\theta + i\sin 3\theta$

$= \text{cis}\, 3\theta$

$= [\text{cis}\, \theta]^3$ {De Moivre}

$= [\cos\theta + i\sin\theta]^3$

$= \cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + [3\cos^2\theta\sin\theta]i - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= [\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta]$

Equating real and imaginary parts,

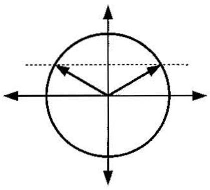
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

and
$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

e Let $x = \sin\theta$

$\therefore\ 3\sin\theta - 4\sin^3\theta = \frac{1}{2}$

$\therefore\ \sin 3\theta = \frac{1}{2}$ {using **d**}



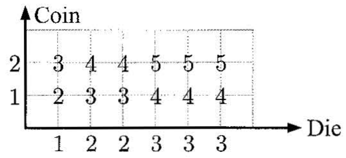
$\therefore\ 3\theta = \begin{cases}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{cases} + k2\pi$

$\therefore\ \theta = \begin{cases}\frac{\pi}{18}\\ \frac{5\pi}{18}\end{cases} + k\frac{12\pi}{18}$

$\therefore\ \theta = \frac{\pi}{18}, \frac{5\pi}{18}, -\frac{7\pi}{18}, \ldots$

$\therefore\ x = \sin\left(\frac{\pi}{18}\right), \sin\left(\frac{5\pi}{18}\right)$, or $\sin\left(-\frac{7\pi}{18}\right)$ are the distinct solutions.

13 **a**



x	2	3	4	5
$P(X = x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	$\frac{3}{12}$

b **i** $P(X = 4) = \frac{5}{12}$

ii $P(X \leqslant 3) = P(X = 2 \text{ or } X = 3)$

$= \frac{1}{12} + \frac{3}{12}$

$= \frac{1}{3}$

c **i** P(one sum is 3 and the other is 4)

= P((Sam a 3 and Pam a 4) or (Sam a 4 and Pam a 3))

$= \frac{3}{12} \times \frac{5}{12} + \frac{5}{12} \times \frac{3}{12}$

$= \frac{30}{144}$

$= \frac{5}{24}$

ii Let Y = the least of the 4 results.

(1) $P(Y = 3) = 0$ {cannot get a 3 with the coin}

(2) To have $Y = 2$, both of the coins must be 2, and both of the dice must be 2 or 3.

$\therefore\ P(Y = 2) = \frac{1}{2} \times \frac{1}{2} \times \frac{5}{6} \times \frac{5}{6}$

$= \frac{25}{144}$

(3) $P(Y = 1) = 1 - \frac{25}{144}$

$= \frac{119}{144}$

(4)

y	1	2	3
$P(Y = y)$	$\frac{119}{144}$	$\frac{25}{144}$	0

$E(Y) = 1 \times \frac{119}{144} + 2 \times \frac{25}{144} + 3 \times 0$

$= \frac{169}{144}$

≈ 1.17

(5) $E(Y^2) = 1^2 \times \frac{119}{144} + 2^2 \times \frac{25}{144} + 3^2 \times 0$

$= \frac{219}{144}$

$\therefore\ \text{Var}(Y) = E(Y^2) - \{E(Y)\}^2$

$= \frac{219}{144} - \left(\frac{169}{144}\right)^2$

≈ 0.143

SOLUTIONS TO TRIAL EXAMINATION 2

NO CALCULATOR

SECTION A

1 **a** $\frac{d}{dx}(x\cos x) = (1)\cos x + x(-\sin x)$

$= \cos x - x\sin x$

$\therefore\ \int(\cos x - x\sin x)\,dx = x\cos x + c$

$\therefore\ \sin x - \int x\sin x\,dx = x\cos x + c$

$\therefore\ \int x\sin x\,dx = \sin x - x\cos x + c$

b **i** Since $f(x)$ is a PDF,

$$\int_0^{\frac{\pi}{2}} f(x)\,dx = 1$$

$$\therefore\ k\int_0^{\frac{\pi}{2}} \sin x\,dx = 1$$

$$\therefore\ k\left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$$

$$\therefore\ k(-\cos\tfrac{\pi}{2} - -1) = 1$$

$$\therefore\ k = 1$$

ii $E(X) = \int_0^{\frac{\pi}{2}} x\,f(x)\,dx$

$= \int_0^{\frac{\pi}{2}} x\sin x\,dx$

$= \left[\sin x - x\cos x\right]_0^{\frac{\pi}{2}}$ {from **a**}

$= (\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}) - (\sin 0 - 0)$

$= 1$

2 **a** The sum of the first n terms of an arithmetic sequence is $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

Now in this case $S_8 = -4$

$\therefore\ \frac{8}{2}(2u_1 + 7d) = -4$

$\therefore\ 2u_1 + 7d = -1$ .... (1)

and also $S_{16} - S_8 = 188$

$\therefore\ \frac{16}{2}(2u_1 + 15d) + 4 = 188$

$\therefore\ 8(2u_1 + 15d) = 184$

$\therefore\ 2u_1 + 15d = 23$ .... (2)

(2) − (1) gives $8d = 24$

$\therefore\ d = 3$

Using (1) we find $u_1 = -11$

$\therefore\ u_n = u_1 + (n-1)d = -11 + 3(n-1)$

$\therefore\ u_n = 3n - 14$

b We need to find n such that $S_n = 25$

$$\therefore \ \frac{n}{2}(2u_1+(n-1)d)=25$$
$$\therefore \ \frac{n}{2}(-22+3(n-1))=25$$
$$\therefore \ -11n+\tfrac{3}{2}n^2-\tfrac{3}{2}n=25$$
$$\therefore \ \tfrac{3}{2}n^2-\tfrac{25}{2}n-25=0$$
$$\therefore \ 3n^2-25n-50=0$$
$$\therefore \ (3n+5)(n-10)=0$$
$$\therefore \ n=10 \quad \{n>0\}$$

Thus we need 10 terms.

3 **a** $$\left(2x-\frac{1}{x}\right)^4$$
$$=(2x)^4+4(2x)^3\left(-\frac{1}{x}\right)+6(2x)^2\left(-\frac{1}{x}\right)^2$$
$$+4(2x)\left(-\frac{1}{x}\right)^3+\left(-\frac{1}{x}\right)^4$$
$$=16x^4-32x^2+24-\frac{8}{x^2}+\frac{1}{x^4}$$

b When $x=1$, $\text{LHS}=(2-1)^4=1$
$$\text{RHS}=16-32+24-8+1$$
$$=1 \ \checkmark$$

c When $x=\sqrt{3}$,
$$(2\sqrt{3}-\tfrac{1}{\sqrt{3}})^4=16(\sqrt{3})^4-32(\sqrt{3})^2+24$$
$$-\frac{8}{(\sqrt{3})^2}+\frac{1}{(\sqrt{3})^4}$$
$$=144-96+24-\tfrac{8}{3}+\tfrac{1}{9}$$
$$=72-2\tfrac{2}{3}+\tfrac{1}{9}$$
$$=69\tfrac{4}{9}$$

4 **a** The lines are coplanar if they intersect.

L_1 meets L_2 where
$$3\lambda+4+1=\frac{\lambda+4-k}{2}=\frac{2\lambda-1-1}{-2}$$
$$\Rightarrow \ 3\lambda+5=\frac{\lambda+4-k}{2}=1-\lambda$$

Now $3\lambda+5=1-\lambda$
$$\Rightarrow \ 4\lambda=-4$$
$$\Rightarrow \ \lambda=-1$$

But $\dfrac{\lambda+4-k}{2}=1-\lambda$ also
$$\therefore \ \frac{-1+4-k}{2}=2$$
$$\therefore \ 3-k=4$$
$$\therefore \ k=-1$$

Using $\lambda=-1$ in L_1, the lines meet at $(1,\ 3,\ -3)$.

b L_1 has direction vector $\begin{pmatrix}3\\1\\2\end{pmatrix}$.

L_2 has direction vector $\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

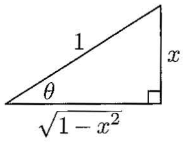
$$\therefore \ \mathbf{p}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&1&2\\1&2&-2\end{vmatrix}$$
$$=\begin{vmatrix}1&2\\2&-2\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&2\\1&-2\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&1\\1&2\end{vmatrix}\mathbf{k}$$
$$=(-2-4)\mathbf{i}-(-6-2)\mathbf{j}+(6-1)\mathbf{k}$$
$$=-6\mathbf{i}+8\mathbf{j}+5\mathbf{k}$$

c Letting $\lambda=0$, the point $(4,\ 4,\ -1)$ on L_1 lies on the plane.

$\therefore$ the equation is
$$-6x+8y+5z=-6(4)+8(4)+5(-1)$$
$$\therefore \ -6x+8y+5z=3$$

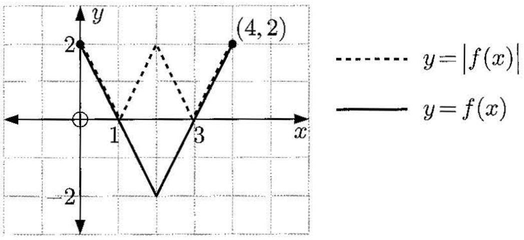
5 **a** $$\int \cos^2\theta\sin\theta\,d\theta$$
$$=-\int[\cos\theta]^2(-\sin\theta)\,d\theta$$
$$=-\frac{[\cos\theta]^3}{3}+c \qquad \{\text{form } -\textstyle\int[f(x)]^n f'(x)\,dx\}$$
$$=-\tfrac{1}{3}\cos^3\theta+c$$

b Let $x=\sin\theta$, so $dx=\cos\theta\,d\theta$
$$\therefore \ \int\frac{x^3}{\sqrt{1-x^2}}\,dx$$
$$=\int\frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$
$$=\int\frac{\sin^3\theta}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$
$$=\int\sin^3\theta\,d\theta$$
$$=\int(1-\cos^2\theta)\sin\theta\,d\theta$$
$$=\int\sin\theta\,d\theta-\int\cos^2\theta\sin\theta\,d\theta$$
$$=-\cos\theta-(-\tfrac{1}{3}\cos^3\theta)+c \qquad \{\text{from } \mathbf{a}\}$$
$$=\tfrac{1}{3}\cos^3\theta-\cos\theta+c$$
$$=\tfrac{1}{3}(1-x^2)^{\frac{3}{2}}-\sqrt{1-x^2}+c$$



$$\sin\theta=\frac{x}{1}$$
$$\therefore \ \cos\theta=\frac{\sqrt{1-x^2}}{1}$$

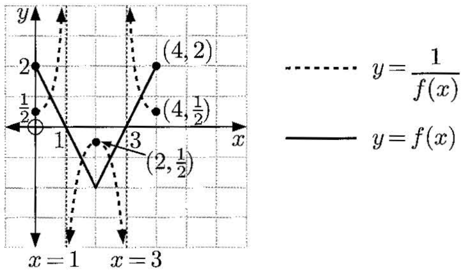
6 **a**



b If $g(x)=\dfrac{1}{f(x)}$, $g(0)=\dfrac{1}{f(0)}=\tfrac{1}{2}$

$\therefore$ the y-intercept of $\dfrac{1}{f(x)}$ is $\tfrac{1}{2}$.

c



d If $[f(x)]^2=1$ then $f(x)=\dfrac{1}{f(x)}$.

These two graphs meet when $x=\tfrac{1}{2},\ 1\tfrac{1}{2},\ 2\tfrac{1}{2}$, or $3\tfrac{1}{2}$.

7 **a** If $\mathrm{P}(X=1)$ is the mean of $\mathrm{P}(X=0)$ and $\mathrm{P}(X=2)$ then

$$\mathrm{P}(X=1)=\frac{\mathrm{P}(X=0)+\mathrm{P}(X=2)}{2}$$

$$\therefore\ 2\,\mathrm{P}(X=1)=\mathrm{P}(X=0)+\mathrm{P}(X=2)$$

$$\therefore\ 2\,\frac{me^{-m}}{1!}=\frac{m^0e^{-m}}{0!}+\frac{m^2e^{-m}}{2!}$$

$$\therefore\ 2me^{-m}=e^{-m}+\tfrac{1}{2}m^2e^{-m}$$

$$\therefore\ e^{-m}(\tfrac{1}{2}m^2-2m+1)=0$$

As $e^{-m}\neq 0,\ \ \tfrac{1}{2}m^2-2m+1=0$

$$\therefore\ m^2-4m+2=0$$

$$\therefore\ m=\frac{4\pm\sqrt{16-4(1)(2)}}{2}$$

$$\therefore\ m=\frac{4\pm 2\sqrt{2}}{2}$$

$$\therefore\ m=2-\sqrt{2}\ \text{ or }\ 2+\sqrt{2}$$

But $m<2$, so $m=2-\sqrt{2}$

b $\mathrm{P}(X\leqslant 1)=\dfrac{m^0e^{-m}}{0!}+\dfrac{m^1e^{-m}}{1!}$

$$=e^{-m}(1+m)$$

$$=e^{\sqrt{2}-2}(3-\sqrt{2})$$

8 **a** $(1+\operatorname{cis}\alpha)(1-\operatorname{cis}\alpha)$

$$=1-\cancel{\operatorname{cis}\alpha}+\cancel{\operatorname{cis}\alpha}-[\operatorname{cis}\alpha]^2$$

$=1-\operatorname{cis}2\alpha$ {De Moivre's theorem}

$$\therefore\ 1+\operatorname{cis}\alpha=\frac{1-\operatorname{cis}2\alpha}{1-\operatorname{cis}\alpha}$$

b $1+\operatorname{cis}\alpha+\operatorname{cis}2\alpha+\operatorname{cis}3\alpha+....+\operatorname{cis}n\alpha$

$=1+\operatorname{cis}\alpha+[\operatorname{cis}\alpha]^2+[\operatorname{cis}\alpha]^3+....+[\operatorname{cis}\alpha]^n$

{De Moivre's theorem}

which is the sum of a geometric series with $u_1=1$ and $r=\operatorname{cis}\alpha$.

Hence, its sum is $\dfrac{u_1(1-r^n)}{1-r}=\dfrac{1(1-[\operatorname{cis}\alpha]^n)}{1-\operatorname{cis}\alpha}$

$$=\frac{1-\operatorname{cis}n\alpha}{1-\operatorname{cis}\alpha}$$

c This is the geometric series in **b** with $n=22$ and $\alpha=\frac{\pi}{11}$

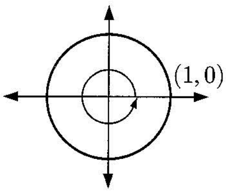
$$\therefore\ 1+\operatorname{cis}\left(\tfrac{\pi}{11}\right)+\operatorname{cis}\left(\tfrac{2\pi}{11}\right)+....+\operatorname{cis}\left(\tfrac{22\pi}{11}\right)$$

$$=\frac{1-\operatorname{cis}\left(\frac{22\pi}{11}\right)}{1-\operatorname{cis}\left(\frac{\pi}{11}\right)}$$

$$=\frac{1-\operatorname{cis}2\pi}{1-\operatorname{cis}\left(\frac{\pi}{11}\right)}$$

$$=\frac{1-1}{1-\operatorname{cis}\left(\frac{\pi}{11}\right)}$$

$$=0$$



9 **a** $\cos\theta$

$$=\frac{\mathbf{a}\bullet\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|}$$

$$=\frac{\mathbf{a}\bullet(|\mathbf{a}|\,\mathbf{b}+|\mathbf{b}|\,\mathbf{a})}{|\mathbf{a}|\,|\mathbf{c}|}$$

$$=\frac{|\mathbf{a}|\,\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|\,\mathbf{a}\bullet\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|}$$

$$=\frac{|\mathbf{a}|\,\mathbf{a}\bullet\mathbf{b}+|\mathbf{b}|\,|\mathbf{a}|^2}{|\mathbf{a}|\,|\mathbf{c}|}$$

$$=\frac{\cancel{|\mathbf{a}|}\,(\mathbf{a}\bullet\mathbf{b}+|\mathbf{a}|\,|\mathbf{b}|)}{\cancel{|\mathbf{a}|}\,|\mathbf{c}|}$$

$$=\frac{\mathbf{a}\bullet\mathbf{b}+|\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$$

b $\cos\phi$

$$=\frac{\mathbf{b}\bullet\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|}$$

$$=\frac{\mathbf{b}\bullet(|\mathbf{a}|\,\mathbf{b}+|\mathbf{b}|\,\mathbf{a})}{|\mathbf{b}|\,|\mathbf{c}|}$$

$$=\frac{|\mathbf{a}|\,\mathbf{b}\bullet\mathbf{b}+|\mathbf{b}|\,\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|\,|\mathbf{c}|}$$

$$=\frac{|\mathbf{a}|\,|\mathbf{b}|^2+|\mathbf{b}|\,\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|\,|\mathbf{c}|}$$

$$=\frac{\cancel{|\mathbf{b}|}\,(|\mathbf{a}|\,|\mathbf{b}|+\mathbf{a}\bullet\mathbf{b})}{\cancel{|\mathbf{b}|}\,|\mathbf{c}|}$$

$$=\frac{|\mathbf{a}|\,|\mathbf{b}|+\mathbf{a}\bullet\mathbf{b}}{|\mathbf{c}|}$$

c From **a** and **b**, $\cos\theta=\cos\phi$

$\therefore$ since ϕ and θ are in $[0,\,\pi]$, $\theta=\phi$.

d Given $\mathbf{a}=\begin{pmatrix}0\\3\\4\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$,

$$|\mathbf{a}|=\sqrt{0+9+16}=5,$$

$$|\mathbf{b}|=\sqrt{4+4+1}=3,$$

and $\mathbf{a}\bullet\mathbf{b}=0-6+4=-2$

$\therefore$ using the result from **c**, vectors which bisect the angle between **a** and **b** have the form

$$\left[5\begin{pmatrix}2\\-2\\1\end{pmatrix}+3\begin{pmatrix}0\\3\\4\end{pmatrix}\right]t,\ \ t\in\mathbb{R}$$

which is $\begin{pmatrix}10t\\-t\\17t\end{pmatrix},\ \ t\in\mathbb{R}.$

SECTION B

10 **a** $\tfrac{1}{2}\cos(A-B)-\tfrac{1}{2}\cos(A+B)$

$$=\tfrac{1}{2}(\cancel{\cos A\cos B}+\sin A\sin B)$$

$$-\tfrac{1}{2}(\cancel{\cos A\cos B}-\sin A\sin B)$$

$$=\tfrac{1}{2}\sin A\sin B+\tfrac{1}{2}\sin A\sin B$$

$$=\sin A\sin B$$

b **i** $\sin 3x\sin x+\sin x\sin x$

$$=\cancel{\tfrac{1}{2}\cos 2x}-\tfrac{1}{2}\cos 4x+\tfrac{1}{2}\cos 0-\cancel{\tfrac{1}{2}\cos 2x}$$

$$=\tfrac{1}{2}-\tfrac{1}{2}\cos 4x$$

$$=\tfrac{1}{2}-\tfrac{1}{2}(1-2\sin^2 2x)$$

$$=\sin^2 2x$$

ii $\sin 5x\sin x+\sin 3x\sin x+\sin x\sin x$

$$=\cancel{\tfrac{1}{2}\cos 4x}-\tfrac{1}{2}\cos 6x+\cancel{\tfrac{1}{2}\cos 2x}-\cancel{\tfrac{1}{2}\cos 4x}$$

$$+\tfrac{1}{2}\cos 0-\cancel{\tfrac{1}{2}\cos 2x}$$

$$=\tfrac{1}{2}-\tfrac{1}{2}\cos 6x$$

$$=\tfrac{1}{2}-\tfrac{1}{2}(1-2\sin^2 3x)$$

$$=\sin^2 3x$$

c **i** $\sin 3x+\sin x=\dfrac{\sin^2 2x}{\sin x}$

ii $\sin 5x+\sin 3x+\sin x=\dfrac{\sin^2 3x}{\sin x}$

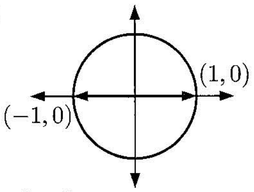
iii $\sin 9x+\sin 7x+\sin 5x+\sin 3x+\sin x$

$=\dfrac{\sin^2 5x}{\sin x}$ {assuming the pattern continues}

d The roots of
$\sin 13x + \sin 11x + ... + \sin 5x + \sin 3x + \sin x = 0$
are the solutions of $\sin 7x = 0$

$\therefore \; 7x = k\pi, \; k \in \mathbb{Z}$

$\therefore \; x = k\frac{\pi}{7}, \; k \in \mathbb{Z}$



$\therefore \; x = -\frac{3\pi}{7}, -\frac{2\pi}{7}, -\frac{\pi}{7}, 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}$

11 **a** If $z = i$,

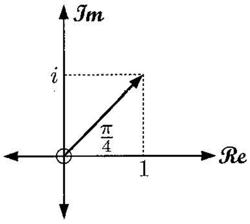
$$w = \frac{i-1}{i+i} = \frac{i-1}{2i}\,\frac{i}{i} = \frac{-1-i}{2i^2} = \frac{-1-i}{-2} = \frac{1+i}{2}$$

$$|w| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}}$$

$$= \sqrt{\tfrac{1}{2}} = \tfrac{1}{\sqrt{2}}$$

and $\arg w = \frac{\pi}{4}$

$\therefore \; w = \frac{1}{\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{4}\right)$



b $w^{14} = \left[\frac{1}{\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{4}\right)\right]^{14}$

$= \left(\frac{1}{\sqrt{2}}\right)^{14} \operatorname{cis}\left(\frac{14 \times \pi}{4}\right)$ {De Moivre's theorem}

$= \frac{1}{2^7} \operatorname{cis}\left(\frac{7\pi}{2}\right)$

$= \frac{1}{128} \operatorname{cis}\left(-\frac{\pi}{2}\right)$

$= \frac{1}{128}(-i)$

$= -\frac{1}{128}i$

c $w = \dfrac{x + iy - 1}{x + iy + i}$

$= \dfrac{(x-1) + iy}{x + i(y+1)} \left[\dfrac{x - i(y+1)}{x - i(y+1)}\right]$

$= \dfrac{x(x-1) + y(y+1) + i[-(x-1)(y+1) + xy]}{x^2 + (y+1)^2}$

$\therefore \; w = \dfrac{(x^2 - x + y^2 + y) + i(-\cancel{xy} - x + y + 1 + \cancel{xy})}{x^2 + (y+1)^2}$

$\therefore \; w = \dfrac{(x^2 - x + y^2 + y) + i(y - x + 1)}{x^2 + (y+1)^2}$

d If $\mathcal{Re}(w) = 1$,

$\cancel{x^2} - x + \cancel{y^2} + y = \cancel{x^2} + (\cancel{y^2} + 2y + 1)$

$\therefore \; -x + y = 2y + 1$

$\therefore \; y = -x - 1, \quad \{x, y\} \neq \{0, -1\}$

e **i** w is real $\Leftrightarrow \; y - x + 1 = 0$

$\Leftrightarrow \; y = x - 1, \quad \{x, y\} \neq \{0, -1\}$

ii w is purely imaginary

$\Leftrightarrow \; x^2 - x + y^2 - y = 0$ and $y - x + 1 \neq 0$

f If $\arg w = \frac{\pi}{4}$,

$x^2 - \cancel{x} + y^2 + \cancel{y} = \cancel{y} - \cancel{x} + 1$

$\Rightarrow \; x^2 + y^2 = 1$

$\Rightarrow \; \sqrt{x^2 + y^2} = 1$

$\Rightarrow \; |z| = 1$

12 **a** L is (0, 2, 2), M is (2, 2, 0), and N is (0, 4, 1).

b $\overrightarrow{\text{ML}} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{\text{MN}} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

c $\overrightarrow{\text{ML}} \times \overrightarrow{\text{MN}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ -2 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 2 \\ -2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} \mathbf{k}$

$= (0 - 4)\mathbf{i} - (-2 + 4)\mathbf{j} + (-4 + 0)\mathbf{k}$

$= -4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$

$= -2(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

d Area of $\triangle$LMN $= \frac{1}{2} \mid \overrightarrow{\text{ML}} \times \overrightarrow{\text{MN}} \mid$

$= \frac{1}{2} \times |-2| \times \sqrt{4 + 1 + 4}$

$= 3$ units^2

e The plane LMN has normal $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and the point (0, 2, 2) lies on the plane.

$\therefore$ the equation of the plane is

$2x + y + 2z = 2(0) + (2) + 2(2)$

which is $2x + y + 2z = 6$

f G is (2, 0, 0) and C is (0, 4, 2).

$\therefore \; \overrightarrow{\text{GC}} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ the line GC has parametric equations
$x = 2 - t, \; y = 2t, \; z = t, \; t \in \mathbb{R}$

$\therefore$ (GC) meets plane LMN where
$2(2 - t) + (2t) + 2(t) = 6$

$\therefore \; 4 - 2t + 2t + 2t = 6$

$\therefore \; 2t = 2$

$\therefore \; t = 1$

$\therefore$ P is (1, 2, 1).

g Given G(2, 0, 0), L(0, 2, 2), and E(0, 4, 0),

$\overrightarrow{\text{GL}} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ and $\overrightarrow{\text{GE}} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$.

If $\text{L}\widehat{\text{G}}\text{E} = \theta$ then

$$\cos\theta = \frac{\overrightarrow{\text{GL}} \bullet \overrightarrow{\text{GE}}}{\mid \overrightarrow{\text{GL}} \mid \mid \overrightarrow{\text{GE}} \mid}$$

$$= \frac{4 + 8 + 0}{\sqrt{4+4+4}\sqrt{4+16+0}}$$

$$= \frac{12}{\sqrt{12}\sqrt{20}}$$

$$= \frac{\sqrt{12}}{\sqrt{20}}$$

$$= \sqrt{\tfrac{3}{5}}$$

$\therefore \; \theta = \arccos\sqrt{\frac{3}{5}}$

$\therefore \; k = \sqrt{\frac{3}{5}}$

13 **a** **i** $x^2y^3 + y^2 = x + k$ passes through (1, 2)

$\therefore \; 8 + 4 = 1 + k$

$\therefore \; k = 11$

ii Differentiating with respect to x gives

$$2xy^3 + x^2 \times 3y^2\,\frac{dy}{dx} + 2y\,\frac{dy}{dx} = 1$$

$$\therefore \; \frac{dy}{dx}(3x^2y^2 + 2y) = 1 - 2xy^3$$

$$\therefore \; \frac{dy}{dx} = \frac{1 - 2xy^3}{3x^2y^2 + 2y}$$

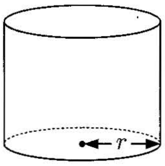
iii At the point $(1, 2)$, $\frac{dy}{dx} = \frac{1-16}{12+4} = \frac{-15}{16}$

$\therefore$ the tangent has equation $\frac{y-2}{x-1} = \frac{-15}{16}$

$\Rightarrow 16y - 32 = -15x + 15$

$\Rightarrow 15x + 16y = 47$

b **i**



$V = \pi r^2 h$ is fixed

$A = \pi r^2 + 2\pi r h$

ii $A = \pi r^2 + 2\cancel{\pi} r\left(\frac{V}{\cancel{\pi} r^2}\right)$

$\therefore A = \pi r^2 + \frac{2V}{r}$

$\therefore A = \pi r^2 + 2Vr^{-1}$

iii $\frac{dA}{dr} = 2\pi r - 2Vr^{-2}$ $\{V$ is a constant$\}$

$= 2\pi r - \frac{2V}{r^2}$

$= 2\pi r - \frac{2\pi r^2 h}{r^2}$

$= 2\pi r - 2\pi h$

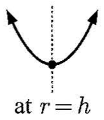
$= 2\pi(r - h)$

$\therefore \frac{dA}{dr} = 0 \Leftrightarrow r = h$

iv $\frac{d^2A}{dr^2} = 2\pi + 4Vr^{-3}$

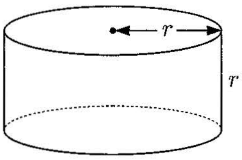
$= 2\pi + \frac{4V}{r^3}$

which is > 0 as $V > 0$ and $r > 0$



$\therefore$ A is a minimum when $r = h$

v The shape should be:



CALCULATOR

SECTION A

1 **a** In $\triangle$BAD, $\text{B}\widehat{\text{D}}\text{A} = 100°$ {angles on a line}

Let $\text{BD} = x$ cm

$\therefore 4^2 = x^2 + 2^2 - 2x2\cos 100°$ {cosine rule}

$\therefore 16 = x^2 + 4 - [4\cos 100°]x$

$\therefore x^2 - [4\cos 100°]x - 12 = 0$

$\therefore x \approx 3.134\,171$ {technology}

$\therefore \text{BD} \approx 3.13$ cm

b Area $\triangle\text{BCD} = \frac{1}{2} \times \text{BD} \times 3 \times \sin 80°$

$\approx 4.629\,83$

$\approx 4.63 \text{ cm}^2$

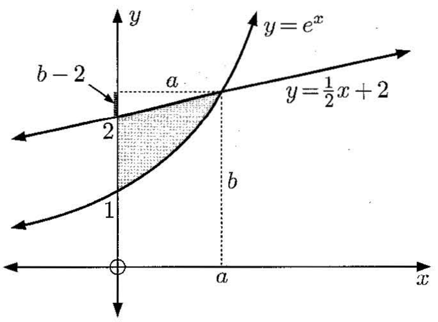
c If h = the length of altitude from B to [DC], then

$\frac{1}{2} \times 3 \times h \approx 4.629\,83$

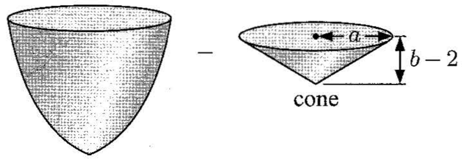
$\therefore h \approx \frac{4.629\,83}{1.5}$

$\therefore h \approx 3.09$ cm

2 **a**



The required volume is



$\therefore V = \pi\int_1^b x^2\,dy - \underbrace{\tfrac{1}{3}\pi a^2(b-2)}_{\text{volume of cone}}$

$\therefore V = \pi\int_1^b (\ln y)^2\,dy - \tfrac{1}{3}\pi a^2(b-2)$

b $y = e^x$ meets $y = \frac{1}{2}x + 2$ where

$e^x = \frac{1}{2}x + 2$

$\Rightarrow e^x - \frac{1}{2}x - 2 = 0$

$\Rightarrow x \approx 0.8951$

$\Rightarrow a \approx 0.8951$

and $\therefore b \approx 2.4475$

c $V \approx \pi\int_1^{2.4475} (\ln y)^2\,dy - \frac{\pi}{3}(0.8951)^2(0.4475)$

$\approx 1.115 \text{ units}^3$

3 **a** P_n is: "$\sum_{i=1}^{n} i(i+4) = \frac{n(n+1)(2n+13)}{6}$" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 5 = 5$

RHS $= \frac{1 \times 2 \times 15}{6} = 5$

$\therefore P_1$ is true.

(2) If P_k is true, then

$$\sum_{i=1}^{k} i(i+4) = \frac{k(k+1)(2k+13)}{6}$$

$$\therefore \sum_{i=1}^{k+1} i(i+4) = \frac{k(k+1)(2k+13)}{6} + (k+1)(k+1+4)$$

$$= \frac{k(k+1)(2k+13)}{6} + \frac{6(k+1)(k+5)}{6}$$

$$= \frac{(k+1)[2k^2 + 13k + 6k + 30]}{6}$$

$$= \frac{(k+1)(2k^2 + 19k + 30)}{6}$$

$$= \frac{(k+1)(k+2)(2k+15)}{6}$$

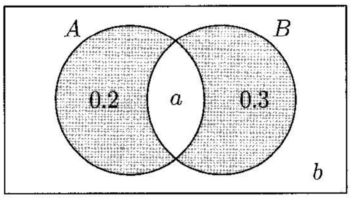
$$= \frac{(k+1)([k+1]+1)(2[k+1]+13)}{6}$$

$\Rightarrow P_{k+1}$ is true.

Thus P_1 is true, and P_{k+1} is true whenever P_k is true. Hence P_n is true. {Principle of mathematical induction}

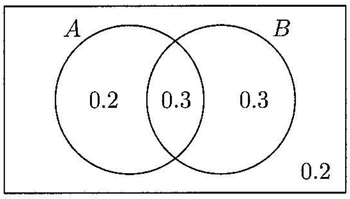
b $$\sum_{i=40}^{60} i(i+4) = \sum_{i=1}^{60} i(i+4) - \sum_{i=1}^{39} i(i+4)$$
$$= \frac{(60)(61)(133)}{6} - \frac{(39)(40)(91)}{6}$$
$$= 57\,470$$

4 **a**



$$\mathrm{P}(A \mid B) = \mathrm{P}(A) \quad \text{for independence}$$
$$\therefore \frac{a}{a+0.3} = a + 0.2$$
$$\therefore a^2 + 0.5a + 0.06 = a$$
$$\therefore a^2 - 0.5a + 0.06 = 0$$
$$\therefore (a-0.2)(a-0.3) = 0$$
$$\therefore a = 0.2 \text{ or } 0.3$$

b The larger a value is $a = 0.3$



i $$\mathrm{P}(B \mid A') = \frac{0.3}{0.3+0.2} = 0.6$$

ii $$\mathrm{P}((A \cup B)') = b = 0.2$$

5 **a**

$$S_4 = \tfrac{3}{4} S_\infty$$
$$\therefore \cancel{u_1}\left(\frac{1-r^4}{\cancel{1-r}}\right) = \tfrac{3}{4}\left(\frac{\cancel{u_1}}{\cancel{1-r}}\right)$$
$$\therefore 1 - r^4 = \tfrac{3}{4}$$
$$\therefore r^4 = \tfrac{1}{4}$$
$$\therefore r = \pm\tfrac{1}{\sqrt{2}}$$

But $r > 0$, so $r = \frac{1}{\sqrt{2}}$

b We need to find n such that

$$S_n > 0.999 S_\infty$$
$$\therefore \cancel{u_1}\left(\frac{1-r^n}{\cancel{1-r}}\right) > 0.999\left(\frac{\cancel{u_1}}{\cancel{1-r}}\right)$$
$$\therefore 1 - r^n > 0.999$$
$$\therefore r^n < 0.001$$
$$\therefore \left(\tfrac{1}{\sqrt{2}}\right)^n < 0.001$$
$$\therefore (\sqrt{2})^n > 1000$$
$$\therefore 2^{\frac{n}{2}} > 1000$$
$$\therefore \log 2^{\frac{n}{2}} > \log 1000$$
$$\therefore \frac{n}{2}\log 2 > \log 1000$$
$$\therefore n > \frac{2\log 1000}{\log 2} \approx 19.93$$
$$\therefore n = 20, 21, 22, ...$$
$$\therefore n \text{ is at least } 20$$

6 **a** The other zeros are complex, and have the form $m \pm ni$ where $m, n \in \mathbb{R}$, $b \neq 0$.

b sum of roots $= \frac{-(-6)}{1} = 6$

$$\therefore -2 + 1 + \cancel{\sqrt{2}} + 1 - \cancel{\sqrt{2}} + m + ni + m - ni = 6$$
$$\therefore 2m = 6$$
$$\therefore m = 3$$

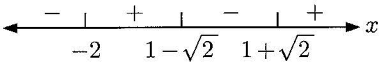
product of roots $= \dfrac{(-1)^5 d}{1} = -d = 20$

$$\therefore -2(1+\sqrt{2})(1-\sqrt{2})(m+ni)(m-ni) = 20$$
$$\therefore -2(1-2)(m^2+n^2) = 20$$
$$\therefore m^2 + n^2 = 10$$
$$\therefore n^2 = 1$$
$$\therefore n = \pm 1$$

$\therefore$ the other two zeros are $3 \pm i$.

c $1 \pm \sqrt{2}$ come from $x^2 - 2x - 1$
$3 \pm i$ come from $x^2 - 6x + 10$
$\therefore p(x) = (x+2)(x^2-2x-1)(x^2-6x+10)$

d $x^2 - 6x + 10$ is always positive



$\therefore p(x) \geqslant 0 \Leftrightarrow -2 \leqslant x \leqslant 1-\sqrt{2}$ or $x \geqslant 1+\sqrt{2}$
Alternatively, we can write
$x \in [-2,\ 1-\sqrt{2}] \cup [1+\sqrt{2},\ \infty[$.

7 **a** The system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 2 & -1 & 1 & 11 \\ 3 & 1 & a & b \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -3 & 3 & -3 \\ 0 & -2 & a+3 & b-21 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & a+3 & b-21 \end{array}\right] \quad R_2 \to -\tfrac{1}{3}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a+1 & b-19 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

b The last equation is $(a+1)z = b - 19$.

So, if $a \neq -1$, then $z = \dfrac{b-19}{a+1}$ is a unique value for z for any fixed b.

$\therefore$ the system would have a unique solution.

c If $a = -1$ and $b = 19$, the last row is all zeros. The system is reduced to two equations in 3 unknowns.

$\therefore$ we have an infinite number of solutions.

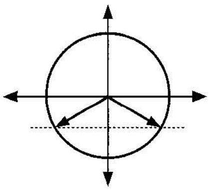
If $z = t$, $y - t = 1 \Rightarrow y = 1 + t$
and $x + 1 + t - t = 7 \Rightarrow x = 6$
$\therefore x = 6$, $y = 1 + t$, $z = t$ for all $t \in \mathbb{R}$.

d The system has no solutions when $a = -1$ and $b \neq 19$.

8 **a** If we square both sides,

$$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = \tfrac{1}{2}$$
$$\therefore 1 + \sin 2\theta = \tfrac{1}{2}$$
$$\therefore \sin 2\theta = -\tfrac{1}{2}$$

b If $\sin 2\theta = -\frac{1}{2}$,



$$2\theta = \left\{ \begin{array}{l} \frac{7\pi}{6} \\ \frac{11\pi}{6} \end{array} \right. + k2\pi$$

$$\therefore \quad \theta = \left\{ \begin{array}{l} \frac{7\pi}{12} \\ \frac{11\pi}{12} \end{array} \right. + k\frac{12\pi}{12}$$

$$\therefore \quad \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12} \qquad \{\theta \in [0, 2\pi]\}$$

c $\cos\left(\frac{7\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) \approx 0.7071 \approx \frac{1}{\sqrt{2}}$ ✓

$\cos\left(\frac{11\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right) \approx -0.7071 \neq \frac{1}{\sqrt{2}}$ ✗

$\cos\left(\frac{19\pi}{12}\right) + \sin\left(\frac{19\pi}{12}\right) \approx -0.7071 \neq \frac{1}{\sqrt{2}}$ ✗

$\cos\left(\frac{23\pi}{12}\right) + \sin\left(\frac{23\pi}{12}\right) \approx 0.7071 = \frac{1}{\sqrt{2}}$ ✓

d Squaring equations can sometimes produce irrelevant solutions.

SECTION B

9 $f(x) = \dfrac{2x-1}{x-3}$

a **i** $f(0) = \frac{-1}{-3} = \frac{1}{3}$

$\therefore$ the y-intercept is $\frac{1}{3}$.

ii $f(x) = 0$ when $2x - 1 = 0$

$\therefore$ the x-intercept is $\frac{1}{2}$.

iii $f(x)$ is undefined when $x - 3 = 0$

$\therefore$ there is a vertical asymptote $x = 3$.

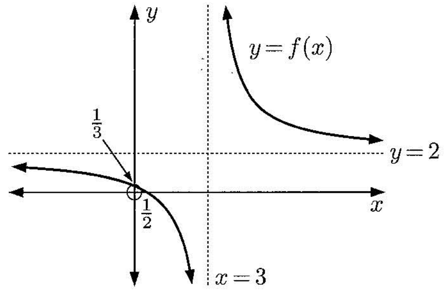
As $x \to \pm\infty$, $f(x) \to 2$

$\therefore$ there is a horizontal asymptote $y = 2$.

b $f(x) = \dfrac{2x-1}{x-3} = \dfrac{2(x-3)+5}{x-3} = 2 + \dfrac{5}{x-3}$

To transform $y = \dfrac{1}{x}$ into $y = f(x)$, we first stretch $y = \dfrac{1}{x}$ vertically with scale factor 5. We then translate the result through $\binom{3}{2}$.

c



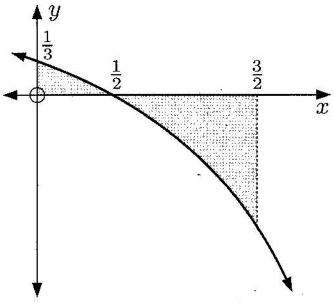
d $f(x) = 2 + \dfrac{5}{x-3}$

$\therefore \quad f'(x) = -\dfrac{5}{(x-3)^2}$

i $f'(x)$ is negative for all $x \neq 3$.

ii $f'(x)$ is undefined when $x = 3$.

e



Shaded area

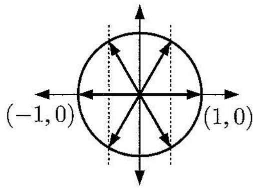
$$= \int_0^{\frac{3}{2}} \left| \frac{2x-1}{x-3} \right| dx$$

$\approx 0.642\,53$ units^2

10 $f(x) = 4\cos^3 x - 3\cos x$

a
$$\begin{aligned} f'(x) &= 12\cos^2 x(-\sin x) - 3(-\sin x) \\ &= -12\cos^2 x \sin x + 3\sin x \\ &= 3\sin x(1 - 4\cos^2 x) \\ &= 3\sin x(1 + 2\cos x)(1 - 2\cos x) \end{aligned}$$

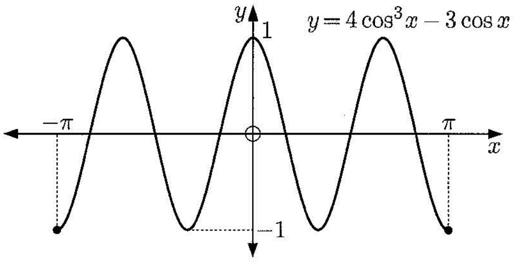
$\therefore \quad f'(x) = 0 \Leftrightarrow \sin x = 0$ or $\cos x = \pm\frac{1}{2}$



$\therefore \quad f'(x) = 0$ when $x = k\frac{\pi}{3}$, $k \in \mathbb{Z}$.

$\therefore \quad f(x)$ has stationary points when $x = k\frac{\pi}{3}$, $k \in \mathbb{Z}$.

b



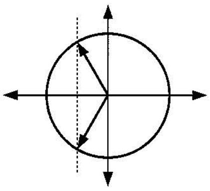
c The graph seems to be that of $y = \cos 3x$ $\{$period $= \frac{2\pi}{3}\}$

d
$$\begin{aligned} \cos 3x &= \cos(2x + x) \\ &= \cos 2x \cos x - \sin 2x \sin x \\ &= (2\cos^2 x - 1)\cos x - 2\sin x \cos x \sin x \\ &= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) \\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \\ &= 4\cos^3 x - 3\cos x \quad \checkmark \end{aligned}$$

e **i** $4t^3 - 3t = -\frac{1}{2}$

$\therefore \quad 4\cos^3 x - 3\cos x = -\frac{1}{2}$ $\{$let $t = \cos x\}$

$\therefore \quad \cos 3x = -\frac{1}{2}$ $\{$using **d**$\}$



$$\therefore \quad 3x = \left\{ \begin{array}{l} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{array} \right. + k2\pi$$

$$\therefore \quad x = \left\{ \begin{array}{l} \frac{2\pi}{9} \\ \frac{4\pi}{9} \end{array} \right. + k\frac{6\pi}{9}$$

$\therefore \quad x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \ldots$

Thus $t = \cos\left(\frac{2\pi}{9}\right)$, $\cos\left(\frac{4\pi}{9}\right)$, or $\cos\left(\frac{8\pi}{9}\right)$ are the only distinct solutions.

ii If $\cos 3x = -\cos x$,

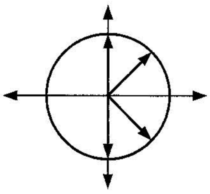
$4\cos^3 x - 3\cos x + \cos x = 0$ $\{$using **d**$\}$

$\therefore \quad 4\cos^3 x - 2\cos x = 0$

$\therefore \quad 2\cos x(2\cos^2 x - 1) = 0$

$\therefore \quad \cos x = 0$ or $\pm\frac{1}{\sqrt{2}}$

$\therefore \quad x = -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$

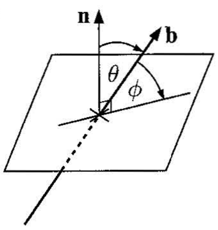


11 **a** The line meets the plane where

$$\begin{aligned} 2(2 - t) + (1 + t) - 3(1 + 3t) &= 22 \\ \therefore \quad 4 - 2t + 1 + t - 3 - 9t &= 22 \\ \therefore \quad -10t + 2 &= 22 \\ \therefore \quad -10t &= 20 \\ \therefore \quad t &= -2 \end{aligned}$$

$\therefore$ they meet at $(4, -1, -5)$.

b



$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\cos\theta = \sin\phi \quad \{\theta + \phi = 90°\}$$

$$\therefore \ \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{b}|}{|\mathbf{n}|\,|\mathbf{b}|}$$

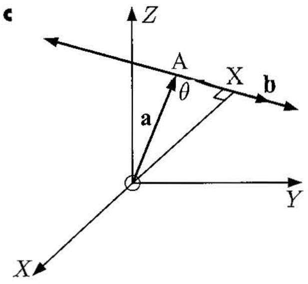
$$= \frac{|-2+1-9|}{\sqrt{4+1+9}\sqrt{1+1+9}}$$

$$= \frac{10}{\sqrt{14}\sqrt{11}}$$

$$= \frac{10}{\sqrt{154}}$$

$\therefore$ the angle between the line and the plane $\phi \approx 53.7°$.

c



Let θ be the angle between $\overrightarrow{\mathrm{OA}}$ and $\mathbf{b}$.

$$\therefore \ \sin\theta = \frac{|\overrightarrow{\mathrm{OX}}|}{|\mathbf{a}|}$$

Thus $|\overrightarrow{\mathrm{OX}}| = |\mathbf{a}| \sin\theta$

Hence
$$|\overrightarrow{\mathrm{OX}}|^2 = |\mathbf{a}|^2 \sin^2\theta$$
$$= |\mathbf{a}|^2 (1 - \cos^2\theta)$$
$$= |\mathbf{a}|^2 - |\mathbf{a}|^2 \cos^2\theta$$
$$= |\mathbf{a}|^2 - \frac{|\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta}{|\mathbf{b}|^2}$$
$$= |\mathbf{a}|^2 - \frac{(\mathbf{a} \bullet \mathbf{b})^2}{|\mathbf{b}|^2}$$
$$\therefore \ |\overrightarrow{\mathrm{OX}}| = \sqrt{|\mathbf{a}|^2 - \frac{(\mathbf{a} \bullet \mathbf{b})^2}{|\mathbf{b}|^2}}$$

d The line with equations $x = 2 - t, \ y = 1 + t, \ z = 1 + 3t, \ t \in \mathbb{R}$ contains the fixed point A(2, 1, 1) and has direction $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

Thus, as $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$,

$$|\overrightarrow{\mathrm{OX}}| = \sqrt{(2^2 + 1^2 + 1^2) - \frac{(-2+1+3)^2}{((-1)^2 + 1^2 + 3^2)}}$$
$$= \sqrt{6 - \tfrac{4}{11}}$$
$$= \sqrt{\tfrac{62}{11}} \text{ units}$$

12 **a** **i**
$$\frac{P(x)}{(x-a)^2} = Q(x) + \frac{bx + c}{(x-a)^2}$$
$$\therefore \ P(x) = Q(x)(x-a)^2 + bx + c$$

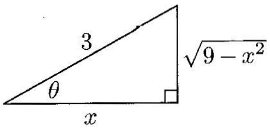
ii
$$P(a) = Q(a) \times 0 + ab + c$$
$$\therefore \ P(a) = ab + c$$
$$P'(x) = Q'(x)(x-a)^2 + Q(x)2(x-a) + b$$
$$\therefore \ P'(a) = 0 + 0 + b$$
$$= b$$

iii Remainder $= bx + c$
$$= P'(a)x + (P(a) - ab)$$
$$= P'(a)x + P(a) - aP'(a)$$
$$= P'(a)(x - a) + P(a)$$

iv For $P(x) = x^5$ divided by $(x+2)^2$,
$$P(-2) = -32$$
and $P'(x) = 5x^4$
$$\therefore \ P'(-2) = 5 \times 16$$
$$= 80$$
$$\therefore \ \text{remainder} = 80(x+2) - 32$$
$$= 80x + 128$$

b Let $x = 3\cos\theta$ so $dx = -3\sin\theta \, d\theta$

$$\therefore \ \int \frac{x}{\sqrt{9 - x^2}} \, dx$$
$$= \int \frac{3\cos\theta}{\sqrt{9 - 9\cos^2\theta}} (-3\sin\theta) \, d\theta$$
$$= \int \frac{3\cos\theta}{3\cancel{\sin\theta}} (-3\cancel{\sin\theta}) \, d\theta$$
$$= \int -3\cos\theta \, d\theta$$
$$= -3\sin\theta + c$$
$$= -3\left(\frac{\sqrt{9 - x^2}}{3}\right) + c$$
$$= -\sqrt{9 - x^2} + c$$



Check:
$$\frac{d}{dx}(-(9 - x^2)^{\frac{1}{2}} + c)$$
$$= -\tfrac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) + 0$$
$$= \frac{x}{\sqrt{9 - x^2}} \quad ✓$$

SOLUTIONS TO TRIAL EXAMINATION 3

NO CALCULATOR

SECTION A

1 **a**
$$\frac{p(x)}{x(2x-3)} = ax + b + \frac{ax + b}{x(2x-3)}$$
$$\therefore \ p(x) = x(ax+b)(2x-3) + ax + b$$
$$\therefore \ p(x) = (ax+b)[x(2x-3) + 1]$$

b Thus
$$p(x) = (ax+b)(2x^2 - 3x + 1)$$
$$= (ax+b)(2x-1)(x-1)$$
$\Rightarrow$ $(2x-1)$ and $(x-1)$ are factors of $p(x)$.

c Since $p(0) = 7$ and $p(2) = 39$,
$b(-1)(-1) = 7$ and $(2a+b)(3)(1) = 39$
$\therefore \ b = 7$ and $2a + 7 = 13$
$\therefore \ 2a = 6$
$\therefore \ a = 3$

Thus,
$$p(x) = (3x+7)(2x^2 - 3x + 1)$$
$$= 6x^3 + 5x^2 - 18x + 7$$

2 **a** Since $\ln\left(\frac{a^2}{b}\right) = k, \ \frac{a^2}{b} = e^k$.

Likewise, since $\ln\left(\frac{b^2}{a^3}\right) = 2, \ \frac{b^2}{a^3} = e^2$.

Thus $a^2 = be^k$ and $a^3 = \frac{b^2}{e^2}$

$\therefore \ a^6 = b^3e^{3k}$ and $a^6 = \frac{b^4}{e^4}$

$\therefore \ \frac{b^4}{e^4} = b^3e^{3k}$ {values for a^6}

$\therefore \ b = e^{3k+4}$

b Since $\dfrac{b^2}{a^3} = e^2$, $a^3 = \dfrac{b^2}{e^2}$

$\therefore\ a^3 = \dfrac{e^{6k+8}}{e^2}$

$\therefore\ a^3 = e^{6k+6}$

$\therefore\ a = \left(e^{6k+6}\right)^{\frac{1}{3}}$

$\therefore\ a = e^{2k+2}$, so $r = 2$ and $s = 2$.

3 **a** Let $\dfrac{x-2}{x^2-1} = \dfrac{a}{x+1} + \dfrac{b}{x-1}$

$= \dfrac{a(x-1)+b(x+1)}{(x+1)(x-1)}$

$= \dfrac{(a+b)x+(b-a)}{x^2-1}$

$\therefore\ a+b = 1$ and $b-a = -2$ {equating coefficients}

Solving these equations simultaneously gives $b = -\frac{1}{2}$ and $a = \frac{3}{2}$.

b $\displaystyle\int_{-4}^{-2} \frac{x-2}{x^2-1}\,dx$

$= \displaystyle\int_{-4}^{-2} \left(\frac{\frac{3}{2}}{x+1} - \frac{\frac{1}{2}}{x-1}\right) dx$

$= \left[\frac{3}{2}\ln|x+1| - \frac{1}{2}\ln|x-1|\right]_{-4}^{-2}$

$= (\frac{3}{2}\ln 1 - \frac{1}{2}\ln 3) - (\frac{3}{2}\ln 3 - \frac{1}{2}\ln 5)$

$= 0 - \frac{1}{2}\ln 3 - \frac{3}{2}\ln 3 + \frac{1}{2}\ln 5$

$= -2\ln 3 + \frac{1}{2}\ln 5$

c $\dfrac{x-2}{x^2-1}$ is undefined when $x = \pm 1$ and $x = 1$ lies in the domain of integration $[1, 3]$.

Thus $\displaystyle\int_1^3 \frac{x-2}{x^2-1}\,dx$ does not exist.

4 **a** Let $u = x$ $\quad v' = \sin 2x$

$u' = 1$ $\quad v = -\frac{1}{2}\cos 2x$

$\therefore\ \int x\sin 2x\,dx$

$= uv - \int u'v\,dx$

$= -\dfrac{x}{2}\cos 2x - \int -\frac{1}{2}\cos 2x\,dx$

$= -\dfrac{x}{2}\cos 2x + \frac{1}{2}(\frac{1}{2}\sin 2x) + c$

$= \frac{1}{4}\sin 2x - \dfrac{x}{2}\cos 2x + c$

b $A = \displaystyle\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

$= \left[\frac{1}{4}\sin 2x - \dfrac{x}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= (\frac{1}{4}\sin\pi - \frac{\pi}{4}\cos\pi) - (\frac{1}{4}\sin 0 - 0)$

$= 0 - \frac{\pi}{4}(-1) - 0$

$= \frac{\pi}{4}$ units^2

$B = -\displaystyle\int_{\frac{\pi}{2}}^{\pi} x\sin 2x\,dx$

$= -\left[\frac{1}{4}\sin 2x - \dfrac{x}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\pi}$

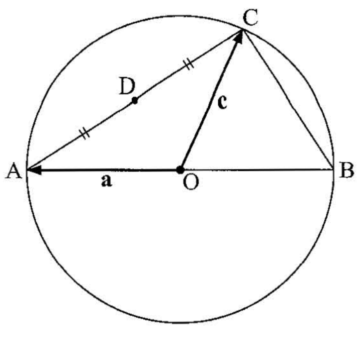
$= -\left[(\frac{1}{4}\sin 2\pi - \frac{\pi}{2}\cos 2\pi) - (\frac{1}{4}\sin\pi - \frac{\pi}{4}\cos\pi)\right]$

$= 0 + \frac{\pi}{2} - 0 + \frac{\pi}{4}$

$= \frac{3\pi}{4}$ units^2

$\therefore\ B = 3A$

5 **a**



$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\mathbf{a} + \mathbf{c}$

$= \mathbf{c} - \mathbf{a}$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \mathbf{a} + \frac{1}{2}\overrightarrow{AC}$

$= \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$= \mathbf{a} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{a}$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c}$

b $\overrightarrow{AC} \bullet \overrightarrow{OD} = (\mathbf{c} - \mathbf{a}) \bullet (\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c})$

$= \cancel{\frac{1}{2}\mathbf{a} \bullet \mathbf{c}} + \frac{1}{2}\mathbf{c} \bullet \mathbf{c} - \frac{1}{2}\mathbf{a} \bullet \mathbf{a} - \cancel{\frac{1}{2}\mathbf{a} \bullet \mathbf{c}}$

$= \frac{1}{2}\left(|\mathbf{c}|^2 - |\mathbf{a}|^2\right)$

$= 0$ as $|\mathbf{a}| = |\mathbf{c}|$ {equal radii}

$\therefore$ $\overrightarrow{AC}$ and $\overrightarrow{OD}$ are perpendicular

$\therefore$ [AC] $\perp$ [OD].

c We have just proved that "the line from the centre of a circle to the midpoint of a chord, is perpendicular to the chord".

6 **a** $g(x)$ is defined when $3 - 2x \geqslant 0$

$\therefore\ -2x \geqslant -3$

$\therefore\ x \leqslant \frac{3}{2}$

b $(g \circ f)(x) = g(f(x))$

$= g(x^2 + 4x)$

$= \sqrt{3 - 2(x^2 + 4x)}$

$\therefore\ (g \circ f)(-3) = \sqrt{3 - 2(9 - 12)}$

$= \sqrt{3 - 2(-3)}$

$= 3$

c $f(x) = x^2 + 4x$ where $x \leqslant -2$ has inverse y where $x = y^2 + 4y$ where $y \leqslant -2$

$\Rightarrow\ y^2 + 4y - x = 0,\ y \leqslant -2$

$\Rightarrow\ y = \dfrac{-4 \pm \sqrt{16 - 4(1)(-x)}}{2},\ y \leqslant -2$

$\Rightarrow\ y = \dfrac{-4 \pm \sqrt{4(4 + x)}}{2},\ y \leqslant -2$

$\Rightarrow\ y = -2 \pm \sqrt{4 + x},\ y \leqslant -2$

$\Rightarrow\ y = -2 - \sqrt{4 + x}$

$\therefore\ f^{-1}(x) = -2 - \sqrt{4 + x},\ x \geqslant -4$

7 **a** $2^{2x} + 2^{x+1} = 15$

Let $m = 2^x$

$\therefore\ m^2 + 2m - 15 = 0$

$\therefore\ (m - 3)(m + 5) = 0$

$\therefore\ m = 3$ or -5

$\therefore\ 2^x = 3$ $\quad$ $\{2^x > 0$ for all $x\}$

$\therefore\ x = \log_2 3$

b $\sin^2 x + \cos x = 1.25$

$\therefore\ 1 - \cos^2 x + \cos x - 1\frac{1}{4} = 0$

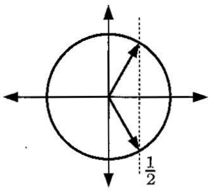
$\therefore\ \cos^2 x - \cos x + \frac{1}{4} = 0$

$\therefore\ 4\cos^2 x - 4\cos x + 1 = 0$

$\therefore\ (2\cos x - 1)^2 = 0$

$\therefore\ \cos x = \frac{1}{2}$

$\therefore\ x = \pm\frac{\pi}{3}$

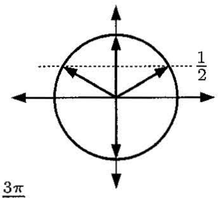


8 $f(x) = \frac{1}{2}\cos 2x + \sin x$

a $f(0) = \frac{1}{2}\cos 0 + \sin 0 = \frac{1}{2}$ $\quad\therefore$ A is $(0,\ \frac{1}{2})$

$f(2\pi) = \frac{1}{2}\cos 4\pi + \sin 2\pi = \frac{1}{2}$ $\quad\therefore$ H is $(2\pi,\ \frac{1}{2})$

b
$$\begin{aligned} f'(x) &= \tfrac{1}{2}(-2\sin 2x) + \cos x \\ &= -2\sin x\cos x + \cos x \\ &= \cos x(1 - 2\sin x) \end{aligned}$$



$$\therefore\ f'(x) = 0 \Leftrightarrow \cos x = 0$$
$$\text{or } \sin x = \tfrac{1}{2}$$
$$\Leftrightarrow x = \tfrac{\pi}{6},\ \tfrac{\pi}{2},\ \tfrac{5\pi}{6},\ \tfrac{3\pi}{2}$$

$\therefore$ B is $(\frac{\pi}{6},\ \frac{3}{4})$, C is $(\frac{\pi}{2},\ \frac{1}{2})$, D is $(\frac{5\pi}{6},\ \frac{3}{4})$, and F is $(\frac{3\pi}{2},\ -\frac{3}{2})$.

c The x-coordinates of E and G are the solutions of $\frac{1}{2}\cos 2x + \sin x = 0$

$$\therefore\ \cos 2x + 2\sin x = 0$$
$$\therefore\ 1 - 2\sin^2 x + 2\sin x = 0$$
$$\therefore\ 2\sin^2 x - 2\sin x - 1 = 0$$
$$\therefore\ \sin x = \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{4}$$
$$\therefore\ \sin x = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

But $-1 \leqslant \sin x \leqslant 1$, so $\sin x = \dfrac{1-\sqrt{3}}{2}$.

At E, $x = \pi - \arcsin\left(\dfrac{1-\sqrt{3}}{2}\right)$.

At G, $x = \arcsin\left(\dfrac{1-\sqrt{3}}{2}\right) + 2\pi$.

SECTION B

9 **a** **i**
$$\begin{aligned} &\cos(A+B) + \cos(A-B) \\ &= \cos A\cos B - \cancel{\sin A\sin B} \\ &\quad + \cos A\cos B + \cancel{\sin A\sin B} \\ &= 2\cos A\cos B \end{aligned}$$

ii
$$\begin{aligned} &\sin(A+B) + \sin(A-B) \\ &= \sin A\cos B + \cancel{\cos A\sin B} \\ &\quad + \sin A\cos B - \cancel{\cos A\sin B} \\ &= 2\sin A\cos B \end{aligned}$$

b **i** Using **a i** with $A = 2\theta,\ B = \theta$

$$\therefore\ \cos(2\theta + \theta) + \cos(2\theta - \theta) = 2\cos 2\theta\cos\theta$$
$$\therefore\ \cos 3\theta + \cos\theta = 2\cos 2\theta\cos\theta$$

ii Likewise using **a ii** with $A = 2\theta,\ B = \theta$

$$\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$$

c $z^2 = [\text{cis}\,\theta]^2 = \text{cis}\,2\theta$

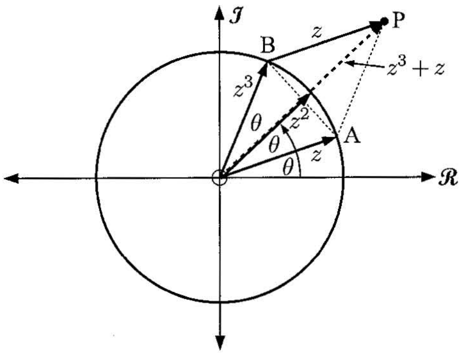
and $z^3 = [\text{cis}\,\theta]^3 = \text{cis}\,3\theta$ {De Moivre's theorem}

d
$$\begin{aligned} z^3 + z &= \text{cis}\,3\theta + \text{cis}\,\theta \\ &= \cos 3\theta + i\sin 3\theta + \cos\theta + i\sin\theta \\ &= [\cos 3\theta + \cos\theta] + i[\sin 3\theta + \sin\theta] \\ &= 2\cos 2\theta\cos\theta + i2\sin 2\theta\cos\theta \\ &= 2\cos\theta[\cos 2\theta + i\sin 2\theta] \\ &= 2\cos\theta\,\text{cis}\,2\theta \end{aligned}$$

e From **d**, $\left|z^3 + z\right| = 2\cos\theta$ and

$\arg(z^3 + z) = 2\theta$

f



$$z^2 = \text{cis}\,2\theta$$
$$\therefore\ \arg(z^2) = 2\theta$$
$$\therefore\ \arg(z^3 + z) = \arg(z^2)$$

g If $z + z^3$ is purely imaginary, then $\overrightarrow{\text{OP}}$ lies on the imaginary axis

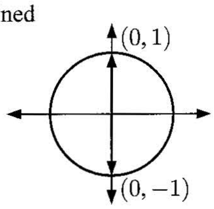
$$\therefore\ 2\theta = \tfrac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$$
$$\therefore\ \theta = \tfrac{\pi}{4} + k\tfrac{\pi}{2},\ k \in \mathbb{Z}$$
$$\therefore\ \theta = \pm\tfrac{\pi}{4} \qquad \{-\tfrac{\pi}{2} \leqslant \theta \leqslant \tfrac{\pi}{2}\}$$

10 **a** **i**
$$\begin{aligned} f(x) &= \tan^3 x \\ \therefore\ f'(x) &= 3\tan^2 x \times \sec^2 x \\ &= 3(\sec^2 x - 1)\sec^2 x \\ &= 3\sec^4 x - 3\sec^2 x \end{aligned}$$

ii Using **i**,

$$\int (3\sec^4 x - 3\sec^2 x)\,dx = \tan^3 x + c$$
$$\therefore\ 3\int \sec^4 x\,dx - 3\int \sec^2 x\,dx = \tan^3 x + c$$
$$\therefore\ 3\int \sec^4 x\,dx - 3\tan x = \tan^3 x + c$$
$$\therefore\ \int \sec^4 x\,dx = \tan x + \tfrac{1}{3}\tan^3 x + c$$

b **i** $y = \sec^2 x = \dfrac{1}{\cos^2 x}$ is undefined when $\cos x = 0$



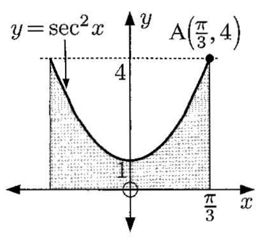
$\Rightarrow$ for the illustrated graph, $a = \frac{\pi}{2}$

ii When $x = \frac{\pi}{3}$, $\cos x = \frac{1}{2}$

$$\therefore\ \sec^2 x = 4$$
$$\therefore\ \text{A is } (\tfrac{\pi}{3},\ 4)$$

c
$$\begin{aligned} \text{Shaded area} &= \int_0^{\frac{\pi}{3}} \sec^2 x\,dx \\ &= \Big[\tan x\Big]_0^{\frac{\pi}{3}} \\ &= \tan\tfrac{\pi}{3} - \tan 0 \\ &= \sqrt{3} \text{ units}^2 \end{aligned}$$

d
$$\begin{aligned} \text{Volume} &= \pi\int_0^{\frac{\pi}{3}} y^2\,dx \\ &= \pi\int_0^{\frac{\pi}{3}} \sec^4 x\,dx \\ &= \pi\Big[\tan x + \tfrac{1}{3}\tan^3 x\Big]_0^{\frac{\pi}{3}} \qquad \{\text{using } \mathbf{a}\} \\ &= \pi(\sqrt{3} + \tfrac{1}{3}(\sqrt{3})^3 - 0) \\ &= \pi(\sqrt{3} + \sqrt{3}) \\ &= 2\pi\sqrt{3} \text{ units}^3 \end{aligned}$$

e 

$$y = \sec^2 x$$
$$\therefore\ \sec x = \sqrt{y} \quad \{\sec x > 0\}$$
$$\therefore\ \cos x = \frac{1}{\sqrt{y}}$$
$$\therefore\ x = \arccos\left(\frac{1}{\sqrt{y}}\right)$$

$$\text{Volume of cylinder} = \pi\left(\tfrac{\pi}{3}\right)^2 4 = \tfrac{4\pi^3}{9}$$

$$\therefore\ \text{volume required} = \tfrac{4\pi^3}{9} - \pi\int_1^4 x^2\,dy = \tfrac{4\pi^3}{9} - \pi\int_1^4 \left[\arccos\left(\frac{1}{\sqrt{y}}\right)\right]^2 dy$$

11 **a** $\mathbf{r} = \begin{pmatrix} 2t+5 \\ -2t-1 \\ t \end{pmatrix}$ is the equation of L.

When $t = 2$, $\mathbf{r} = \begin{pmatrix} 9 \\ -5 \\ 2 \end{pmatrix}$ $\therefore$ $(9, -5, 2)$ lies on L.

b $\mathbf{n} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$ and the plane has equation

$$3x - 4y - z = 3(-1) - 4(0) - (4)$$

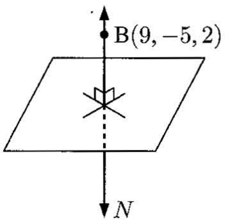
which is $3x - 4y - z = -7$

c The line $x = 2t + 5$, $y = -2t - 1$, $z = t$ meets the plane when $3(2t+5) - 4(-2t-1) - t = -7$

$$\therefore\ 6t + 15 + 8t + 4 - t = -7$$
$$\therefore\ 13t = -26$$
$$\therefore\ t = -2$$

Hence $x = 2(-2) + 5 = 1$,
$y = -2(-2) - 1 = 3$,
$z = -2$

The line meets the plane at $(1, 3, -2)$.

d 

Line N has equation
$x = 9 + 3s$, $y = -5 - 4s$,
$z = 2 - s$, $s \in \mathbb{R}$.

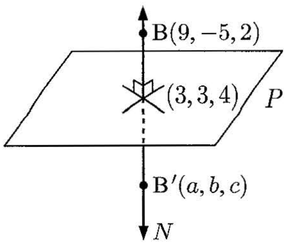
or $\mathbf{r}_2 = \begin{pmatrix} 9+3s \\ -5-4s \\ 2-s \end{pmatrix}$

e Line N meets P where

$$3(9+3s) - 4(-5-4s) - (2-s) = -7$$
$$\therefore\ 27 + 9s + 20 + 16s - 2 + s = -7$$
$$\therefore\ 26s + 45 = -7$$
$$\therefore\ 26s = -52$$
$$\therefore\ s = -2$$

Hence $x = 9 + 3(-2) = 3$
$y = -5 - 4(-2) = 3$
$z = 2 - (-2) = 4$

$\therefore$ line N meets the plane at $(3, 3, 4)$.

f 

If B' is (a, b, c),

$$\frac{a+9}{2} = 3, \quad \frac{b-5}{2} = 3, \quad \frac{c+2}{2} = 4$$
$$\therefore\ a + 9 = 6, \quad b - 5 = 6, \quad c + 2 = 8$$
$$\therefore\ a = -3, \quad b = 11, \quad c = 6$$

$\therefore$ B' is $(-3, 11, 6)$

g Given $C(1, 3, -2)$ and $B'(-3, 11, 6)$,

$$\overrightarrow{CB'} = \begin{pmatrix} -4 \\ 8 \\ 8 \end{pmatrix} = -4(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$

$\Rightarrow$ $\overrightarrow{CB'}$ is parallel to $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.
$\{\overrightarrow{CB'}$ is a constant multiple of $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\}$

12 **a** P_n is: "$2^{4n+3} + 3^{3n+1}$ is divisible by 11" for all $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $2^{4n+3} + 3^{3n+1} = 2^7 + 3^4$
$= 128 + 81$
$= 209$
$= 11 \times 19$

$\therefore$ P_1 is true.

(2) If P_k is true then
$2^{4k+3} + 3^{3k+1} = 11A$ where $A \in \mathbb{Z}^+$.

Now $2^{4(k+1)+3} + 3^{3(k+1)+1}$
$= 2^{4k+4+3} + 3^{3k+3+1}$
$= 2^4 2^{4k+3} + 3^3 3^{3k+1}$
$= 16(11A - 3^{3k+1}) + 27 \times 3^{3k+1}$
$= 16 \times 11 \times A - 16 \times 3^{3k+1} + 27 \times 3^{3k+1}$
$= 16 \times 11 \times A + 11 \times 3^{3k+1}$
$= 11(16A + 3^{3k+1})$ where $16A + 3^{3k+1} \in \mathbb{Z}^+$

Thus P_{k+1} is true.

So, P_1 is true, and P_{k+1} is true provided P_k is true for all $k \geqslant 1$.

Hence, P_n is true.
{Principle of mathematical induction}

b Since a, b, c are consecutive terms in an arithmetic sequence, $b - a = c - b$

$\therefore$ $a + c = 2b$

But $a + b + c = 33$, so $2b + b = 33$
$\therefore$ $3b = 33$
$\therefore$ $b = 11$

Since a, $b + 1$, $c + 29$ are consecutive terms in a geometric sequence,

$$\frac{b+1}{a} = \frac{c+29}{b+1}$$

$$\therefore\ \frac{12}{a} = \frac{c+29}{12} \quad \text{where } a + c = 22$$

$$\text{Thus} \quad \frac{12}{a} = \frac{22 - a + 29}{12}$$

$$\therefore \quad \frac{12}{a} = \frac{51 - a}{12}$$

$$\therefore \quad 144 = 51a - a^2$$

$$\therefore \quad a^2 - 51a + 144 = 0$$

$$\therefore \quad (a-3)(a-48) = 0$$

$$\therefore \quad a = 3 \text{ or } 48$$

$$\therefore \quad a = 3, \quad b = 11, \quad c = 19 \quad \text{or}$$

$$a = 48, \quad b = 11, \quad c = -26$$

CALCULATOR

SECTION A

1 **a** **i** $z = r \operatorname{cis} \theta$

$$z^3 = (r \operatorname{cis} \theta)^3$$

$$= r^3 \operatorname{cis} 3\theta \qquad \{\text{De Moivre's theorem}\}$$

ii $\sqrt[3]{z} = z^{\frac{1}{3}}$

$$= (r \operatorname{cis} \theta)^{\frac{1}{3}}$$

$$= r^{\frac{1}{3}} \operatorname{cis}\left(\tfrac{\theta}{3}\right) \qquad \{\text{De Moivre's theorem}\}$$

b $-11 + ai = (1 - ai)^3$

$$= 1 - 3(ai) + 3(ai)^2 - (ai)^3$$

$$= 1 - 3ai - 3a^2 + a^3 i$$

$$= (1 - 3a^2) + (a^3 - 3a)i$$

$$\Rightarrow \quad 1 - 3a^2 = -11 \quad \text{and} \quad a^3 - 3a = a$$

{equating real and imaginary parts}

$$\therefore \quad 3a^2 = 12 \quad \textbf{and} \quad a^3 - 4a = 0$$

$$\therefore \quad a^2 = 4 \quad \textbf{and} \quad a(a^2 - 4) = 0$$

$$\therefore \quad a = \pm 2$$

2 **a** If $f(x) = \begin{cases} \frac{1}{2}e^{-bx}, & 0 \leqslant x \leqslant 1, \quad b \neq 0 \\ 0, & \text{otherwise} \end{cases}$

is a well defined PDF, then

$$\int_0^1 \tfrac{1}{2}e^{-bx}\, dx = 1 \qquad \{f(x) > 0 \text{ always}\}$$

$$\therefore \quad \left[\frac{1}{-2b}e^{-bx}\right]_0^1 = 1$$

$$\therefore \quad e^{-b} - e^0 = -2b$$

$$\therefore \quad e^{-b} + 2b - 1 = 0$$

b Using technology, $b \approx -1.256\,431$

$$\therefore \quad b \approx -1.256$$

c $\mu = \text{E}(X)$

$$\approx \int_0^1 x\left(\tfrac{1}{2}e^{1.256\,431x}\right) dx$$

$$\approx 0.602 \qquad \{\text{using technology}\}$$

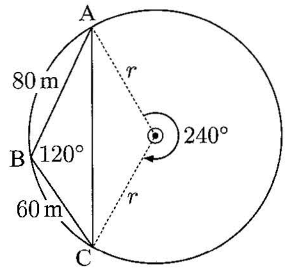
d $\text{Var}(X)$

$$= \text{E}(X^2) - \{\text{E}(X)\}^2$$

$$\approx \int_0^1 x^2\left(\tfrac{1}{2}e^{1.256\,431x}\right) dx - \mu^2$$

$$\approx 0.0772 \qquad \{\text{using technology}\}$$

3 **a**



Using the cosine rule in $\triangle$ABC,

$$\text{AC}^2 = 60^2 + 80^2 - 2(60)(80)\cos 120^\circ$$

$$\therefore \quad \text{AC}^2 = 60^2 + 80^2 + 60 \times 80$$

$$\therefore \quad \text{AC}^2 = 14\,800$$

b Reflex $\text{A}\widehat{\text{O}}\text{C} = 240^\circ$ {angle at the centre theorem}

$$\therefore \quad \text{A}\widehat{\text{O}}\text{C} = 120^\circ$$

In $\triangle$AOC, $\text{AC}^2 = r^2 + r^2 - 2rr\cos 120^\circ$ {cosine rule}

$$\therefore \quad \text{AC}^2 = 2r^2 + r^2$$

$$\therefore \quad \text{AC}^2 = 3r^2$$

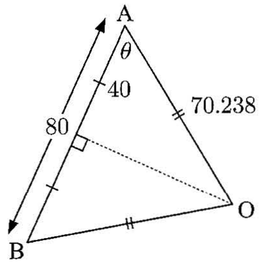
c From **a** and **b**, $3r^2 = 14\,800$

$$\therefore \quad r^2 \approx 4933.33....$$

$$\therefore \quad r \approx 70.238....$$

$$\therefore \quad r \approx 70.2$$

d



$$\cos\theta \approx \frac{40}{70.238}$$

$$\theta \approx \arccos\left(\frac{40}{70.238}\right)$$

$$\therefore \quad \text{B}\widehat{\text{A}}\text{O} \approx 55.3^\circ$$

4 **a** When no replacement occurs, we do not have a repetition of n independent trials each with the same probability of success. However, since n is very large, the probability of a success each time will be almost the same.

b $X \sim \text{B}(20,\ 0.032)$

i $\text{P}(X \leqslant 2)$

$$= \text{P}(X = 0,\ 1, \text{ or } 2)$$

$$= \tbinom{20}{0}(0.032)^0(0.968)^{20} + \tbinom{20}{1}(0.031)^1(0.968)^{19} + \tbinom{20}{2}(0.032)^2(0.968)^{18}$$

$$\approx 0.975$$

ii $\text{P}(X \geqslant 4)$

$$= 1 - \text{P}(X \leqslant 3)$$

$$\approx 0.003\,37 \qquad \{\text{using technology}\}$$

5 **a** $\dfrac{d}{dx}(x^{n+1}\ln x) = (n+1)x^n \ln x + x^{n+1}\left(\dfrac{1}{x}\right)$

$$= (n+1)x^n \ln x + x^n$$

$$\text{Thus} \quad \int [(n+1)x^n \ln x + x^n]\, dx = x^{n+1}\ln x + c$$

$$\therefore \quad (n+1)\int x^n \ln x\, dx + \frac{x^{n+1}}{n+1} = x^{n+1}\ln x + c$$

provided $n \neq -1$

$$\therefore \quad \int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

$$= \frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right) + c$$

provided $n \neq -1$

b When $n = -1$,

$$\int x^n \ln x \, dx = \int \frac{\ln x}{x} \, dx$$

$$= \int (\ln x)^1 \left(\frac{1}{x}\right) dx$$

This has the form $\int (f(x))^n f'(x) \, dx$, so

$$\int x^n \ln x \, dx = \frac{(\ln x)^2}{2} + c$$

6 $s = t\sin\left(\frac{t}{2}\right) + 2\cos\left(\frac{t}{2}\right)$ metres

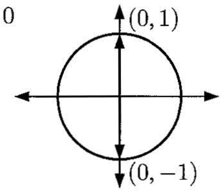
a $v = \frac{ds}{dt} = (1)\sin\left(\frac{t}{2}\right) + t\left(\frac{1}{2}\right)\cos\left(\frac{t}{2}\right) + 2\left(\frac{1}{2}\right)\left(-\sin\left(\frac{t}{2}\right)\right)$

$\therefore \ v = \cancel{\sin\left(\frac{t}{2}\right)} + \frac{t}{2}\cos\left(\frac{t}{2}\right) - \cancel{\sin\left(\frac{t}{2}\right)}$

$\therefore \ v = \frac{t}{2}\cos\left(\frac{t}{2}\right) \ \text{m s}^{-1}$

b The particle is at rest when $v = 0$

$\therefore \ \frac{t}{2}\cos\left(\frac{t}{2}\right) = 0$



$\therefore \ t = 0$ or $\frac{t}{2} = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$

$\therefore \ t = 0$ or $t = \pi + k2\pi$

At $t = \pi$, $\cos\left(\frac{t}{2}\right)$ changes sign.

$\therefore$ the particle reverses direction at $t = \pi$ seconds.

Its position at this time is $s(\pi) = \pi$ m right of O.

c $a = \frac{dv}{dt} = \frac{1}{2}\cos\left(\frac{t}{2}\right) + \frac{t}{2}\left(-\sin\left(\frac{t}{2}\right)\right)\frac{1}{2}$

$\therefore \ a(t) = \frac{1}{2}\cos\left(\frac{t}{2}\right) - \frac{t}{4}\sin\left(\frac{t}{2}\right)$

$\therefore \ a\left(\frac{\pi}{3}\right) = \frac{1}{2}\cos\left(\frac{\pi}{6}\right) - \frac{\pi}{12}\sin\left(\frac{\pi}{6}\right)$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\pi}{12} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{4} - \frac{\pi}{24} \ \text{m s}^{-2}$

7 **a** $\arctan\left(\frac{x}{3}\right) + \arctan 6 = \arctan 3$

Let $\alpha \quad + \quad \beta \quad = \quad \theta$

where $\tan\alpha = \frac{x}{3}$, $\tan\beta = 6$, $\tan\theta = 3$.

Now $\tan(\alpha + \beta) = \tan\theta$

$\therefore \ \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan\theta$

$\therefore \ \frac{\frac{x}{3} + 6}{1 - \frac{x}{3}(6)} = 3$

$\therefore \ \frac{x}{3} + 6 = 3(1 - 2x)$

$\therefore \ x + 18 = 9 - 18x$

$\therefore \ 19x = -9$

$\therefore \ x = -\frac{9}{19}$

b $y = \arctan\left(\frac{x}{3}\right)$

$\therefore \ \frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{3}\right)^2} \times \frac{1}{3}$

$= \frac{1}{1 + \frac{x^2}{9}} \times \frac{1}{3}$

$= \frac{3}{9 + x^2}$

8 **a** If $x = 0$, $y = 3$, $\quad 9 + 3b + c = 0 \quad \ldots\ldots (1)$
If $x = 2$, $y = -1$, $\quad 5 + 2a - b + c = 0 \quad \ldots\ldots (2)$
If $x = 8$, $y = 7$, $\quad 113 + 8a + 7b + c = 0 \quad \ldots\ldots (3)$

b
$$\left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 2 & -1 & 1 & -5 \\ 0 & 3 & 1 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 0 & -11 & 3 & 93 \\ 0 & 3 & 1 & -9 \end{array}\right] \quad R_2 \to 4R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 8 & 7 & 1 & -113 \\ 0 & 3 & 1 & -9 \\ 0 & 0 & 20 & 180 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ R_3 \to 3R_2 + 11R_3 \end{array}$$

c From row 3, $20c = 180 \ \therefore \ c = 9$

From row 2, $3b + 9 = -9 \ \therefore \ 3b = -18$

$\therefore \ b = -6$

From row 1, $8a + 7(-6) + 9 = -113$

$\therefore \ 8a - 42 + 9 = -113$

$\therefore \ 8a = -80$

$\therefore \ a = -10$

$\therefore \ a = -10$, $b = -6$, and $c = 9$

SECTION B

9 Let X = result of a student in the Science exam.

$X \sim \text{N}(56.7, 18.2^2)$

a **i** P(result between 65 and 85 inclusive)

$= \text{P}(65 \leqslant X \leqslant 85)$

≈ 0.264

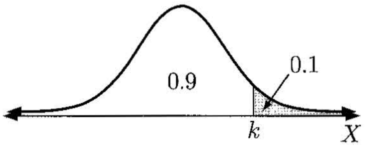
ii P(result of at least 70)

$= \text{P}(X \geqslant 70)$

≈ 0.232

b **i** We need to find k where

$\text{P}(X \leqslant k) = 0.9$

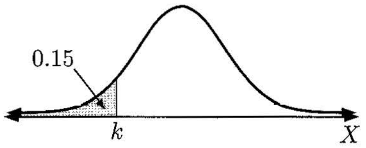


$\therefore \ k \approx 80$

$\therefore$ a result of 80 or more receives an 'A'.

ii We need to find k where

$\text{P}(X \leqslant k) = 0.15$



$\therefore \ k \approx 37.8$

$\therefore$ a result of 37 or less receives an 'F'.

c

Student	(1)	(2)	(3)	(4)
	A	A	F	F
	A	F	A	F
	A	F	F	A
	F	F	A	A
	F	A	F	A
	F	A	A	F

There are six different permutations of 2 'A's and 2 'F's.

$\therefore$ P(two 'A's and two 'F's)

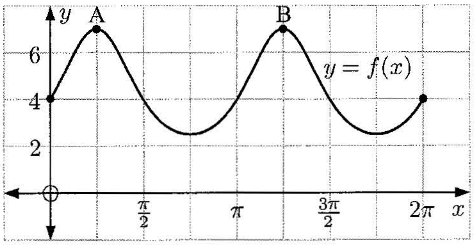
$= 6 \times (0.1)^2 \times (0.15)^2$

$= 0.001\,35$

d $Y \sim \mathrm{B}(20,\ 0.1)$

i $\mathrm{P}(Y=0)$ ≈ 0.122 **ii** $\mathrm{P}(Y \geqslant 3) = 1 - \mathrm{P}(Y \leqslant 2)$ ≈ 0.323

10 **a** **i** $f(x) = \dfrac{12 + 2\sin 2x}{3 - \sin 2x}$

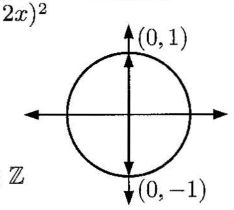


ii $f'(x)$

$$= \frac{4\cos 2x(3 - \sin 2x) - (12 + 2\sin 2x)(-2\cos 2x)}{(3 - \sin 2x)^2}$$

$$= \frac{12\cos 2x - \cancel{4\sin 2x\cos 2x} + 24\cos 2x + \cancel{4\sin 2x\cos 2x}}{(3 - \sin 2x)^2}$$

$$= \frac{36\cos 2x}{(3 - \sin 2x)^2}$$



$\therefore\ f'(x) = 0$

$\Leftrightarrow\ \cos 2x = 0$

$\Leftrightarrow\ 2x = \frac{\pi}{2} + k\pi,\ \ k \in \mathbb{Z}$

$\Leftrightarrow\ x = \frac{\pi}{4} + k\frac{\pi}{2}$

$\Leftrightarrow\ x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

$\therefore$ the local maximum at A is $(\frac{\pi}{4},\ 7)$ and the local maximum at B is $(\frac{5\pi}{4},\ 7)$.

b **i** $h'(x)$

$$= \frac{(2b\cos 2x)(c - \sin 2x) - (a + b\sin 2x)(-2\cos 2x)}{(c - \sin 2x)^2}$$

$$= \frac{2bc\cos 2x - \cancel{2b\sin 2x\cos 2x} + 2a\cos 2x + \cancel{2b\sin 2x\cos 2x}}{(c - \sin 2x)^2}$$

$$= \frac{2\cos 2x(a + bc)}{(c - \sin 2x)^2}$$

ii $h'(x) = 0 \ \Leftrightarrow\ x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$ {as in **b**}

$\therefore\ M = h\left(\frac{\pi}{4}\right) = \dfrac{a + b(1)}{c - 1} = \dfrac{a + b}{c - 1}$

and $m = h\left(\frac{3\pi}{4}\right) = \dfrac{a + b(-1)}{c - (-1)} = \dfrac{a - b}{c + 1}$

iii $H = M - m = \dfrac{a + b}{c - 1} - \dfrac{a - b}{c + 1}$

$$= \frac{(a + b)(c + 1) - (a - b)(c - 1)}{(c - 1)(c + 1)}$$

$$= \frac{\cancel{ac} + a + bc + \cancel{b} - \cancel{ac} + a + bc - \cancel{b}}{c^2 - 1}$$

$$= \frac{2a + 2bc}{c^2 - 1}$$

$$= \frac{2(a + bc)}{c^2 - 1}$$

iv When $a = 12,\ b = 2,\ c = 3,$

$H = \dfrac{2(12 + 6)}{9 - 1} = \dfrac{2 \times 18}{8} = \frac{9}{2}$

Now $f\left(\frac{3\pi}{4}\right) = \dfrac{12 + 2\sin\left(\frac{3\pi}{2}\right)}{3 - \sin\left(\frac{3\pi}{2}\right)}$

$$= \frac{12 + 2(-1)}{3 - (-1)} = \frac{10}{4} = \frac{5}{2}$$

and $7 - \frac{5}{2} = \frac{9}{2}$, so the result is verified.

11 **a** L_1: $\mathbf{r} = \begin{pmatrix} -1 + 2\lambda \\ -\lambda \\ 1 + 2\lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

L_2: $\mathbf{s} = \begin{pmatrix} 1 + 3\mu \\ 1 - \mu \\ 2 + 2\mu \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

b The lines intersect if $\mathbf{r} = \mathbf{s}$

$\therefore\ -1 + 2\lambda = 1 + 3\mu$ .... (1)

$-\lambda = 1 - \mu$ .... (2)

$1 + 2\lambda = 2 + 2\mu$ .... (3)

From (2), $\lambda = \mu - 1$

So, in (1) $-1 + 2\mu - 2 = 1 + 3\mu$

$\therefore\ \mu = -4$ and $\lambda = -5$

However, in (3) $1 + 2(-5) = -9$ and $2 + 2(-4) = -6$

$\therefore$ (3) is not satisfied by the solutions from (1) and (2).

$\therefore$ L_1 and L_2 do not intersect.

Also, $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ L_1 and L_2 are not parallel.

Thus L_1 and L_2 are skew lines.

c A vector perpendicular to L_1 **and** L_2 is

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & -1 & 2 \end{vmatrix}$$

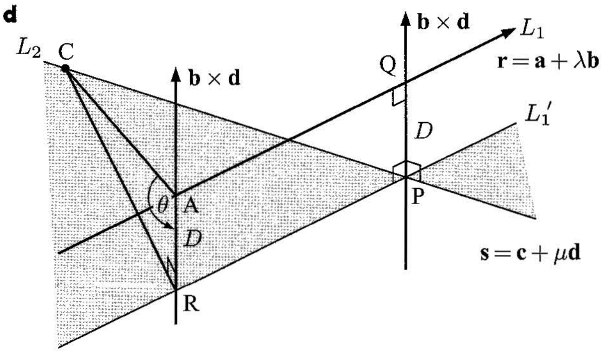
$$= \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix}\mathbf{k}$$

$$= (-2 + 2)\mathbf{i} - (4 - 6)\mathbf{j} + (-2 + 3)\mathbf{k}$$

$$= 0\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}$$

$$= 2\mathbf{j} + \mathbf{k}$$

d



L_1 is translated to L_1' meeting L_2 at P.

i $\mathbf{b} \times \mathbf{d}$ is perpendicular to L_1 and L_2, so $\mathbf{b} \times \mathbf{d}$ is a normal to the shaded plane containing L_1 and L_2.

ii $\mathrm{A\widehat{R}C} = 90°$ {AR is normal to the shaded plane}

Let $\mathrm{C\widehat{A}R} = \theta$ be the angle between $\overrightarrow{\mathrm{AC}}$ and $\mathbf{b} \times \mathbf{d}$.

$\therefore\ \cos\theta = \dfrac{\mathrm{AR}}{\mathrm{AC}} = \dfrac{D}{|\overrightarrow{\mathrm{AC}}|}$

$\therefore\ D = |\overrightarrow{\mathrm{AC}}|\cos\theta$

$$= \frac{|\mathbf{c} - \mathbf{a}|\,|\mathbf{b} \times \mathbf{d}|\cos\theta}{|\mathbf{b} \times \mathbf{d}|}$$

$$= \frac{|(\mathbf{c} - \mathbf{a}) \bullet (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$$

e For the original lines L_1 and L_2,

$$\mathbf{a} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{c} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{b} \times \mathbf{d} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \{\text{from } \mathbf{c}\}$$

$$\therefore \quad D = \frac{\left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right|}{\sqrt{0+4+1}} = \tfrac{3}{\sqrt{5}} \text{ units}$$

12 **a** $v_x = v_0 \cos\theta$

$\therefore \quad x = \int v_0 \cos\theta \, dt$

$\therefore \quad x = (v_0 \cos\theta)t + c$

But when $t = 0, \ x = 0$

$\therefore \quad c = 0$

$\therefore \quad x = (v_0 \cos\theta)t \quad .... \ (1)$

Likewise, $v_y = v_0 \sin\theta - gt$

$\therefore \quad y = \int (v_0 \sin\theta - gt) \, dt$

$\therefore \quad y = (v_0 \sin\theta)t - g\dfrac{t^2}{2} + d$

When $t = 0, \ y = 0$

$\therefore \quad d = 0$

$\therefore \quad y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \quad .... \ (2)$

b From (1), $t = \dfrac{x}{v_0 \cos\theta}$

$$\therefore \quad y = v_0 \sin\theta \left(\frac{x}{v_0 \cos\theta} \right) - \tfrac{1}{2} g \left(\frac{x}{v_0 \cos\theta} \right)^2$$

$$\therefore \quad y = (\tan\theta)x - \tfrac{1}{2} g \frac{x^2}{v_0^2 \cos^2\theta}$$

$$\therefore \quad y = (\tan\theta)x - (\sec^2\theta) \frac{gx^2}{2v_0^2}$$

c The path has equation $y = ax + bx^2$ which is a quadratic in x.

$\therefore$ the path is parabolic.

Furthermore, since $b < 0$, the path is concave.

d
$$\frac{dy}{dx} = \tan\theta - \sec^2\theta \, \frac{2gx}{2v_0^2}$$

$$\therefore \quad \frac{dy}{dx} = 0 \quad \Leftrightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{gx}{v_0^2 \cos^2\theta}$$

$$\Leftrightarrow \quad x = \frac{v_0^2 \sin\theta \cos\theta}{g}$$

$$\Leftrightarrow \quad x = \frac{v_0^2 2 \sin\theta \cos\theta}{2g}$$

$$\Leftrightarrow \quad x = \frac{v_0^2 \sin 2\theta}{2g}$$

The maximum height is

$$= \left(\frac{\sin\theta}{\cancel{\cos\theta}} \right) \left(\frac{v_0^2 \sin\theta \cancel{\cos\theta}}{g} \right) - \tfrac{1}{2} g \, \frac{v_0^4 \sin^2\theta \cancel{\cos^2\theta}}{g^2 v_0^2 \cancel{\cos^2\theta}}$$

$$= \frac{v_0^2 \sin^2\theta}{g} - \frac{1}{2} \frac{v_0^2 \sin^2\theta}{g}$$

$$= \frac{v_0^2 \sin^2\theta}{2g}$$

e Range $= 2\left(\dfrac{v_0^2 \sin 2\theta}{2g} \right) = \dfrac{v_0^2 \sin 2\theta}{g}$ metres.

f The range is maximised when $\sin 2\theta = 1$

$\therefore \quad 2\theta = 90°$

$\therefore \quad \theta = 45°$

g **i** The maximum height $= \dfrac{300^2 \left(\frac{1}{\sqrt{2}} \right)^2}{2 \times 9.81}$ m

≈ 2294 m

ii Range $= \dfrac{300^2 \times 1}{9.81}$

≈ 9174 m

h Using **e**, $\quad 9500 = \dfrac{400^2 \sin 2\theta}{g}$

$\therefore \quad \sin 2\theta \approx 0.582\,47$

$\therefore \quad 2\theta \approx 35.62°$

$\therefore \quad \theta \approx 17.8°$

The angle is about 17.8°.